ACKERMANN FUNCTIONS AND TRANSFINITE ORDINALS

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Abstract—A set of binary operators are defined and shown to be equivalent to Ackermann functions. The same set of operators are used to develop a notation for writing the sequence of transfinite ordinals.

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1. ACKERMANN FUNCTIONS

Ackermann functions are examples of nonprimitive recursive functions which grow so fast with respect to their arguments, that it becomes almost impossible to imagine the *order* of the magnitudes involved[1]. It is easy to give the equations for these functions, but writing the solutions is by no means straightforward. As shown here, one way to alleviate this difficulty is to define a set of binary operators and write the functions in terms of these operators. While the inherent complexity of the functions has to remain, these operators do provide a notation for writing the functions explicitly. Of the many versions of Ackermann functions, the one given below is perhaps the simplest. The functions can be defined by the equations

$$\begin{aligned} A(0,n) &= bn \\ A(k,1) &= b \\ A(k,n) &= A[k-1, A(k,n-1)]. \end{aligned}$$

It is easy to write the first few functions explicitly, but only the first few.

$$\begin{aligned} A(1,n) &= A[0, A(1,n-1)] = bA(1,n-1) = b^2A(1,n-2) \\ &= b^{n-1}A(1,1) = b^n. \\ A(2,n) &= A[1, A(2,n-1)] = b^{A(2,n-1)} \\ &= b^{b^{\cdot^{\cdot^{b}}}}, \text{ the total number of } b'\text{s tilting forward being } n \\ &= {}^n b, \text{ read as } b \textit{ tetrated to } n. \\ A(3,n) &= A[2, A(3,n-1)] \\ &= {}^{b^{\cdot^{\cdot^{b}}}}b, \text{ the total number of } b'\text{s tilting backward being } n. \end{aligned}$$

While the above can be read as b *pentated to* n, it becomes quite clear that we have reached an impasse in writing $A(4,n)$, and $A(k,n)$ in general. We overcome this difficulty by defining *Ackermann operators*, symbolized as $\otimes^k$, and showing that these operators are nothing but the Ackermann functions. The operators are defined with respect to a specific base b. In the following, the product

$$[b \otimes^k [b \otimes^k [\cdots [b \otimes^k b] \cdots]]]$$

is simply written as

$$b \otimes^k b \otimes^k \cdots b \otimes^k \; b$$

to make the notation less cluttered. The operator $\otimes^0$ is defined as the usual multiplication.

$$\begin{aligned}
b \; \otimes^0 \; n &= \underbrace{b \; + \; b \; + \; \cdots \; + \; b}_{n} = bn \\
b \; \otimes^k \; 1 &= b \\
b \; \otimes^1 \; n &= \underbrace{b \; \otimes^0 \; b \; \otimes^0 \; \cdots \; \otimes^0 \; b}_{n} = b^n \\
b \; \otimes^2 \; n &= \underbrace{b \; \otimes^1 \; b \; \otimes^1 \; \cdots \; \otimes^1 \; b}_{n} = {}^n b \\
& \cdots \quad \cdots \quad \cdots \\
b \; \otimes^k \; n &= \underbrace{b \; \otimes^h \; b \; \otimes^h \; \cdots \; \otimes^h \; b}_{n}, \quad \text{where } h = k - 1.
\end{aligned}$$

It is visibly clear from the above that

$$\begin{aligned}
b \; \otimes^0 \; n &= bn \\
b \; \otimes^k \; 1 &= b \\
b \; \otimes^k \; n &= b \; \otimes^h \; [b \; \otimes^k \; (n-1)].
\end{aligned}$$

Hence, we can write

$$A(k, n) = b \; \otimes^k \; n.$$

It is possible to write the other versions of Ackermann function in terms of the operators $\otimes^k$, even though not as simply as the above. The operator $\otimes^0$ is nothing but the ordinary multiplication and the operator $\otimes^1$ is the usual exponentiation. Continuing in the same vein, we may call $\otimes^2$ as *tetration*, $\otimes^3$ as *pentation*, and so on. It should be clear that Ackermann functions are not just ad hoc functions furnishing queer examples of nonprimitive functions, but provide an infinite number of operators, of which only the first two are commonly used.

2. TRANSFINITE ORDINALS

A sophisticated application of the notation we have developed occurs when we consider the transfinite ordinals of Cantor. The difficulty in listing the transfinite ordinals may be judged from the following description given by Halmos[2].

> ... In this way we get successively ω, $\omega 2$, $\omega 3$, $\omega 4$, $\cdots$. An application of the axiom of substitution yields something that follows them all in the same sense in which ω follows the natural numbers; that something is ω^2. After that the whole thing starts over again: $\omega^2 + 1$, $\omega^2 + 2$, $\cdots$, $\omega^2 + \omega$, $\omega^2 + \omega + 1$, $\omega^2 + \omega + 2$, $\cdots$, $\omega^2 + \omega 2$, $\omega^2 + \omega 2 + 1$, $\cdots$, $\omega^2 + \omega 3$, $\cdots$, $\omega^2 + \omega 4$, $\cdots$, $\omega^2 2$, $\cdots$, $\omega^2 3$, $\cdots$, ω^3, $\cdots$, ω^4, $\cdots$, ω^ω, $\cdots$, $\omega^{(\omega^\omega)}$, $\cdots$, $\omega^{(\omega^{(\omega^\omega)})}$, $\cdots$. The next one after all this is ϵ_0; then come $\epsilon_0 + 1$, $\epsilon_0 + 2$, $\cdots$, $\epsilon_0 + \omega$, $\cdots$, $\epsilon_0 + \omega 2$, $\cdots$, $\epsilon_0 + \omega^2$, $\cdots$, $\epsilon_0 + \omega^\omega$, $\cdots$, $\epsilon_0 2$, $\cdots$, $\epsilon_0 \omega$, $\cdots$, $\epsilon_0 \omega^\omega$, $\cdots$, ϵ_0^2, $\cdots \cdots \cdots$.

An inspection of the ordinals appearing in the explanation above shows that the *significant* ordinals are ω^2, ω^ω, $\epsilon_0, \cdots$. It is easy to see that in our notation the entire sequence of significant transfinite ordinals can be written as

$$\omega \otimes^0 \omega,\ \omega \otimes^1 \omega,\ \omega \otimes^2 \omega,\ \omega \otimes^3 \omega, \cdots$$

The fact that the Ackermann functions and the transfinite ordinals can be written in terms of the operator $\otimes^k$ clearly shows that it is a complex operator. Some idea of the complexity can be obtained if one attempts to calculate the number represented by $2 \otimes^3 4$. A detailed evaluation shows that

$$2 \otimes^3 4 = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}},\ \text{the total number of } 2's \text{ tilting forward being } 65536.$$

References

1. E. Mendelson, *Introduction to Mathematical Logic,* D. Van Nostrand Company, New York, NY, (1979).
2. P.R. Halmos, *Naive Set Theory,* D. Van Nostrand Company, New York, NY, (1960).