Hall's Theorem and Compound Matrices

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Abstract—Compound matrices which list the entire set of matchings in a bigraph are used to prove the theorem of Hall.

Keywords—Compound matrices; Hall's theorem; Matchings.

1. INTRODUCTION

The purpose of this paper is to show that the theory of compound matrices [1] provides a convenient and efficient notation for stating and proving Hall's theorem [2]. The theorem states that in a bigraph with vertex sets V_1 and V_2, a complete matching of V_1 into V_2 exists, if and only if, every subset of k vertices in V_1 is adjacent to at least k vertices in V_2, for all values of k. In the bigraph, we assume $|V_1| = m$, $|V_2| = n$ and $m \leq n$.

2. COMPOUND MATRICES

From a marix $\mathbf{A}$ of order $m \times n$, when the minors of order k are arranged in the lexical order, the resulting $\binom{m}{k} \times \binom{n}{k}$ matrix is called the k^{th} compound of $\mathbf{A}$ and written as $\mathbf{A}^{(k)}$. See below where the adjacency matrix $\mathbf{A}$ of the bigraph in Fig. 1 is taken as an example.

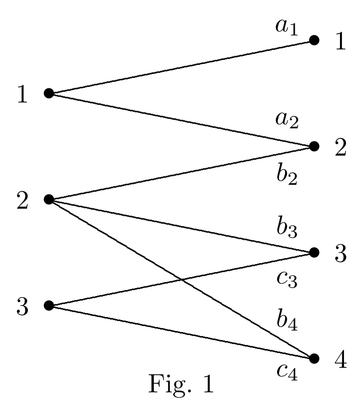

Fig. 1

$$\mathbf{A}^{(1)} = \mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} a_1 & a_2 & 0 & 0 \\ 0 & b_2 & b_3 & b_4 \\ 0 & 0 & c_3 & c_4 \end{pmatrix} \end{array}$$

$$\mathbf{A}^{(2)} = \begin{array}{c} \\ 12 \\ 13 \\ 23 \end{array} \begin{array}{c} \begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \end{array} \\ \begin{pmatrix} a_1b_2 & a_1b_3 & a_1b_4 & a_2b_3 & a_2b_4 & 0 \\ 0 & a_1c_3 & a_1c_4 & a_2c_3 & a_2c_4 & 0 \\ 0 & 0 & 0 & b_2c_3 & b_2c_4 & b_3c_4 - b_4c_3 \end{pmatrix} \end{array}$$

$$\mathbf{A}^{(3)} = \begin{array}{c} \\ 123 \end{array} \begin{array}{cccc} 123 & 124 & 134 & 234 \\ \big(a_1b_2c_3 & a_1b_2c_4 & a_1b_3c_4 - a_1b_4c_3 & a_2b_3c_4 - a_2b_4c_3 \big) \end{array}$$

Given below are two theorems on compound matrices that we need for proving Hall's theorem. Here **P** and **Q** are assumed to be conformable and **R** is of order r.

$$(\mathbf{PQ})^{(k)} = \mathbf{P}^{(k)}\mathbf{Q}^{(k)}$$
$$|\mathbf{R}^{(k)}| = |\mathbf{R}|^{\binom{r-1}{k-1}}$$

With these preliminaries we can now deal with Hall's theorem.

3. PROOF OF HALL'S THEOREM

It is easy to state Hall's theorem in terms of the adjacency matrix **A** of the bigraph.

Hall's Theorem. *For some k, a row of $\mathbf{A}^{(k)} = 0$, if and only if, $\mathbf{A}^{(m)} = 0$.*

Proof. We carry out the proof in two parts.

Part 1.

$$\begin{aligned} \text{For some } k,\ \text{a row of } \mathbf{A}^{(k)} = 0 &\Rightarrow |\mathbf{A}^{(k)}\mathbf{A}_T^{(k)}| = 0 \\ &\Rightarrow |(\mathbf{A}\mathbf{A}_T)^{(k)}| = 0 \\ &\Rightarrow |\mathbf{A}\mathbf{A}_T|^{\binom{m-1}{k-1}} = 0 \\ &\Rightarrow |\mathbf{A}\mathbf{A}_T| = 0 \\ &\Rightarrow \mathbf{A}^{(m)}\mathbf{A}_T^{(m)} = 0 \\ &\Rightarrow \mathbf{A}^{(m)} = 0 \end{aligned}$$

Part 2.

$$\begin{aligned} \mathbf{A}^{(m)} = 0 &\Rightarrow \text{For } k = m,\ \text{the row matrix } \mathbf{A}^{(k)} = 0 \\ &\Rightarrow \text{For some } k,\ \text{a row of } \mathbf{A}^{(k)} = 0 \end{aligned}$$

Hall's theorem immediately follows.

4. CONCLUSION

Since the terms in a determinant represent matchings, it is not surprising that compound matrices are useful in analyzing matching problems. It is easy to see that the compound matrices give the *entire* set of matchings in a bigraph.

REFERENCES

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2. N. Deo, *Graph Theory,* Prentice Hall of India, New Delhi, India, (1984).