# Intuitive Set Theory

K. K. Nambiar
6664 Autumn Glen Drive, West Chester, OH 45069

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**Abstract**—A set theory called Intuitive Set Theory is introduced in which Skolem Paradox does not appear. A measure function called real measure is defined in which Axiom of Choice cannot produce a nonmeasurable set.

**Keywords**—Skolem Paradox; Bonded set; Real measure.

## 1. INTRODUCTION

*Intuitive Set Theory* (IST) defined here is an extension of the Real Set Theory (RST) given in an earlier paper [1], in the sense that the Axiom of Fusion defined below holds good in IST, in addition to all the axioms of RST. The concept of a *class of bonded sets* introduced here allows us to construct a model [2] of IST with *real cardinality* $\aleph_0$, thereby avoiding the Skolem Paradox. The notion of a bonded set is useful also for defining a *real measure* in which the Axiom of Choice (AC) cannot be used to produce a nonmeasurable set.

## 2. INTUITIVE SET THEORY

Some definitions are given below to facilitate our discussion. The crucial concept here is a bonded set, a set which even the axiom of choice cannot penetrate to choose an element.

*Bonded Set:* A set is a bonded set, if the AC can choose only the set itself as a whole and not its constituent elements.
*Class:* A set which has sets as its elements.
*Bonded Class:* A class which has only bonded sets as its elements.
$2^{\aleph_\alpha}$*:* Power set of $\aleph_\alpha$.
$\binom{\aleph_\alpha}{\aleph_\alpha}$*: Real power set* of $\aleph_\alpha$, defined as the class of all the subsets of $\aleph_\alpha$ of cardinality $\aleph_\alpha$.
*Real Cardinality:* Cardinality of a bonded class, taking the bonded sets in it as elements.
*R:* The class of *infinite* recursive subsets of positive integers, a class of cardinality $\aleph_0$.
*x:* An element of $R$, which defines an infinite binary sequence and hence equivalent to a real point in the interval $(0, 1]$.
$(x]$*:* The same as the cartesian product $x \times 2^{\aleph_\alpha}$, which we will call the *infinitesimal* $x$.
*Microcosm:* The same as the cartesian product $R \times 2^{\aleph_\alpha}$, considered an adequate representation of all the points of $(0, 1]$.
*N:* An element of $R$, which defines an infinite binary sequence written leftwards and hence called a *supernatural number.*
$[N)$*:* The same as the cartesian product $2^{\aleph_\alpha} \times N$, and hence called a *cosmic stretch.*
*Macrocosm:* The same as the cartesian product $2^{\aleph_\alpha} \times R$, considered an adequate representation of all *counting numbers*, even those above supernatural numbers.

Using these definitions, we can now state the axiom that is central to IST.

Axiom of Fusion. $(0, 1] = \binom{\aleph_\alpha}{\aleph_\alpha} = R \times 2^{\aleph_\alpha}$, *where* $x \times 2^{\aleph_\alpha}$ *is a bonded set.*

Thus the axiom of fusion says that the *significant* part of the power set of $\aleph_\alpha$ consists of $\aleph_0$ bonded sets with cardinality $2^{\aleph_\alpha}$. We accept the axiom of fusion as part of IST.

## 3. INFINITESIMAL GRAPH

Note, as an example, that the infinite sequence $.010 * * * * \cdots$ can be used to represent the interval $(.25, .375]$, if we accept certain assumptions about the representation:

The initial binary string, $.010 = .25$, represents the initial point of the interval.
The length of the binary string, 3 in our case, decides the length of the interval as $2^{-3} = .125$.
Every $*$ in the infinite $*$-string can be substituted by a 0 or 1, to create $2^{\aleph_\alpha}$ points in the interval.

We will accept the fact that a *nonterminating* binary sequence, $.bbbb \cdots$, or equivalently, an infinite recursive subset of positive integers, can be used *uniquely* to represent a real number in the interval $(0, 1]$. A slight generalization of these ideas allows us to draw a graph to represent $[0, 1]$.

To visualize the definitions given above and also to aid our intuition, we define a graph we call $G_{\aleph_0}$. In the Cantorian tradition, we define the *infinitesimal* graph $G_{\aleph_0}$ as the infinite sequence of graphs shown as Figures $\{G_1, G_3, G_7, \cdots\}$. Note that the graph $G_k$ has $k$ nodes between the nodes 0 and $\aleph_0$, labelled 1 to $k$. We will take it as axiomatic that the graph $G_{\aleph_0}$ shown in Figure $G_{\aleph_0}$ has $\aleph_0$ nodes and $\aleph_0$ edges, and also that the edges and nodes of the graph represent respectively the infinitesimals $(x]$ and real points $x$ defined earlier. In the graphs, nodes are unconventionally drawn as vertical lines for clarity, and also in deference to Dedekind whose *cut* it represents.

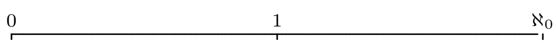


Fig. $G_1$

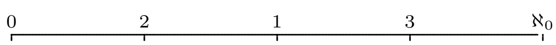


Fig. $G_3$

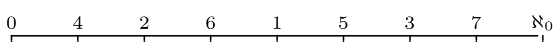


Fig. $G_7$

⋮


Fig. $G_{\aleph_0}$

Even though $G_{\aleph_0}$ has no role to play with the formal part of our arguments here, it can aid considerably in the visualization of the concepts introduced.

## 4. SKOLEM PARADOX

Cantor's theorem asserts that every model of Zermelo-Fraenkel set theory (ZF) has to have cardinality greater than $\aleph_0$. On the other hand, Löwenheim-Skolem theorem (LS) says that there is a model of ZF theory, whose cardinality is $\aleph_0$. These two statements together is called Skolem Paradox.

Intuitive set theory provides a reasonable way to resolve the Skolem Paradox. We merely take the LS theorem as stating that the *real* cardinality of a model of IST need not be greater than $\aleph_0$. Clearly, the Upward Löwenheim-Skolem theorem also cannot raise any paradox in IST.

## 5. REAL MEASURE

In measure theory, it is known that there are sets which are not Lebesgue measurable, but it has not been possible to date to construct such a set, without invoking the axiom of choice. The usual method is to choose exactly one element from each of the set $x \times 2^{\aleph_\alpha}$ we defined earlier, and show that the set thus created is not Lebesgue measurable. This method is obviously not possible in IST, since $x \times 2^{\aleph_\alpha}$ is a bonded set. Noting this fact, we define *real measure* in IST to be the same as Lebesgue measure, except that the universal set we consider is restricted to the interval $[0, 1]$. If we accept the axiom of fusion, it is easy to see that the axiom of choice cannot be used to produce a nonmeasurable subset of $[0, 1]$. Thus, it would not be unreasonable to assert that there are no sets in IST which are not real measurable.

## 6. CONCLUSION

From the definitions given in the beginning, it should be obvious that there is a one-to-one correspondence between the points in $(0, 1]$ and the counting numbers. Hence, our statements about microcosm are equally applicable to macrocosm also. Further, it should be clear that intuitive set theory will suffice for scientists to investigate the phenomenal world, and real set theory will be needed only by mathematicians who want to probe the complexities of the noumenal universe.

## REFERENCES

1. K. K. Nambiar, *Real Set Theory,* Computers and Mathematics, Vol. ??, No. ?, pp. ?-?, (199?).

2. P. Benacerraf, and H. Putnam, eds., *Philosophy of Mathematics: Selected Readings,* Prentice-Hall, Englewood Cliffs, NJ, (1964).