Matchings and Pfaffians

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Abstract—Pfaffian of a skew symmetric matrix obtained from an undirected graph is used to enumerate the macthings in the graph.

Keywords—Undirected graph; Matchings; Pfaffian.

1. INTRODUCTION

In an undirected graph [1], a set of edges is called a *matching*, if no two edges are adjacent. We shall call a matching *exhaustive*, if every node has an edge of the matching incident on it. A graph is called *simple* if it has no loops and no parallel edges. In this paper, we consider only simple graphs and exhaustive matchings. The purpose of this paper is to demonstrate that pfaffians [2] can be used to enumerate the exhaustive matchings in a graph.

2. MATCHINGS

To illustrate the use of pfaffians, consider the graph shown in Fig. 1 as an example. Fig. 2 shows the graph redrawn with two directed labelled edges replacing each undirected edge. Note that the labels on the oppositely directed edges are of opposite signs.

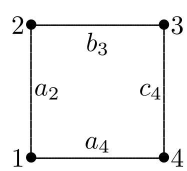

Fig. 1

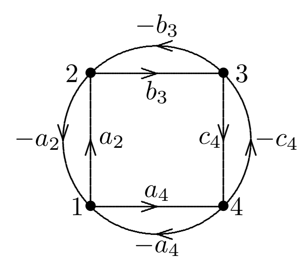

Fig. 2

The adjacency matrix of this *derived graph* can be written as

$$\mathbf{A} = \begin{bmatrix} 0 & a_2 & 0 & a_4 \\ -a_2 & 0 & b_3 & 0 \\ 0 & -b_3 & 0 & c_4 \\ -a_4 & 0 & -c_4 & 0 \end{bmatrix}$$

Since $\mathbf{A}$ is a skew symmetric matrix, the determinant Δ has to be a perfect square, as shown below.

$$\begin{aligned} \Delta &= -a_2 \begin{vmatrix} a_2 & b_3 & 0 \\ 0 & 0 & c_4 \\ -a_4 & c_4 & 0 \end{vmatrix} - a_4 \begin{vmatrix} -a_2 & 0 & b_3 \\ 0 & -b_3 & 0 \\ -a_4 & 0 & -c_4 \end{vmatrix} \\ &= a_2 c_4 \begin{vmatrix} a_2 & b_3 \\ -a_4 & c_4 \end{vmatrix} + a_4 b_3 \begin{vmatrix} -a_2 & b_3 \\ -a_4 & -c_4 \end{vmatrix} \\ &= (a_2 c_4 + a_4 b_3)^2 \end{aligned}$$

The square root $a_2c_4 + a_4b_3$ is called the pfaffian of the matrix $\mathbf{A}$. For our example, it is easy to see that the pfaffian enumerates the exhaustive matchings, the two matchings in the graph are $\{a_2, c_4\}$ and $\{a_4, b_3\}$. We shall now give a direct method to get the pfaffian of a matrix.

The expansion of a pfaffian is very similar to the expansion of a determinant around a row. The difference in the case of a pfaffian is only that the cofactor of x_i is obtained by deleting the two columns and the two rows containing both x_i and $-x_i$, instead of just the row and the column containing x_i. The following illustrates the procedure for getting the pfaffian Π.

$$\begin{aligned}\Pi &= -a_2 \begin{vmatrix} 0 & c_4 \\ -c_4 & 0 \end{vmatrix}^* - a_4 \begin{vmatrix} 0 & b_3 \\ -b_3 & 0 \end{vmatrix}^* \\ &= a_2c_4 + a_4b_3\end{aligned}$$

Here the * superscript indicates that it is the pfaffian that we are evaluating and not the determinant.

Nothing substantially different happens if we consider graphs with more nodes. Hence, we can make the following general statement.

THEOREM. *The exhaustive matchings in an undirected graph are enumerated by the pfaffian of the adjacency matrix of the derived graph.*

PROOF. Note that we have labelled the graph in Fig. 2 in a special way. In labelling the edges, the nodes $1, 2, 3, 4$ are also visualized as nodes a, b, c, d respectively. Then the edge going from the lower node 2 to the higher node 3 is designated as b_3, and similarly all the other edges. This way we can recognize which nodes are matched by which edges without referring to the graph. It is known that in each term of the pfaffian, each node occurs exactly once. Thus, each term has to represent an exhaustive matching.

3. CONCLUSION

It is well-known that the determinant of the adjacency matrix of a directed graph gives the matchings in the corresponding bigraph. What we have shown here is that in the case of an undirected graph, the matchings are given by the pfaffian of the adjacency matrix of a derived graph.

REFERENCES

1. N. Deo, *Graph Theory,* Prentice Hall of India, New Delhi, India, (1984).
2. A. C. Aitken, *Determinants and Matrices,* Oliver and Boyd, London, (1956).