Real Set Theory

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Abstract—A set theory called Real Set Theory is defined in which Generalized Continuum Hypothesis and Axiom of Choice hold good.

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1. INTRODUCTION

Real Set Theory (RST) defined here is an amalgamation of the three axiomatic theories [1] *Elementary Arithmetic* (EA) of Gödel, *Zermelo-Fraenkel Set Theory* (ZF), and first-order *Predicate Calculus with Equality* (PC). All the three have been slightly enriched and respectively called *Extended Elementary Arithmetic* (EEA), *Augmented Zermelo-Fraenkel Theory* (AZF), and *Enhanced Predicate Calculus* (EPC). A significant fact about RST is that *Generalized Continuum Hypothesis* (GCH)is a theorem in it, and hence the *Axiom of Choice* (AC) also. The purpose of this paper is to explain the theories EEA, AZF, and EPC.

2. EXTENDED ELEMENTARY ARITHMETIC

The two binary operators used in the Elementary Arithmetic are the usual $+$ and $\times$. We extend these [2] to an infinite sequence of operators $\otimes^k$, using the multiplication operator $\times$ as a basis.

$$\begin{aligned} &m \otimes^0 n = mn, \\ &m \otimes^k 1 = m, \\ &m \otimes^k n = m \otimes^h [m \otimes^h [\cdots [m \otimes^h m]]], \end{aligned}$$

where the number of m's in the product is n and $h = k - 1$. It is easy to see that

$$\begin{aligned} &m \otimes^1 n = m^n, \\ &m \otimes^2 n = m^{m^{\cdot^{\cdot^{m}}}}, \end{aligned}$$

where the number of m's tilting forward is n. We can continue to expand the operators in this fashion further, but we will not do so, since it does not serve any purpose in what follows. We use these operators for symbolizing the transfinite cardinals of Cantor.

The definition for ω_1, the ordinal corresponding to $\aleph_1$, is usually given as

$$\begin{aligned} \omega_1 &= \{0, 1, 2, \cdots \omega, \cdots \omega^2, \cdots \omega^\omega, \cdots {}^\omega\omega, \cdots, \cdots, \cdots\} \\ &= \{0, 1, 2, \cdots \omega, \cdots \omega \otimes^0 \omega, \cdots \omega \otimes^1 \omega, \cdots \omega \otimes^2 \omega, \cdots \omega \otimes^3 \omega, \cdots, \cdots, \cdots\}. \end{aligned}$$

Taking a clue from here, we conclude that in our notation, $\aleph_1$ can be written as

$$\aleph_1 = \aleph_0 \otimes^{\aleph_0} \aleph_0$$

and, in general,

$$\aleph_{\alpha+1} = \aleph_\alpha \otimes^{\aleph_0} \aleph_\alpha.$$

Looking at the definition of $\otimes^1$, it is easy to see that

$$2^{\aleph_\alpha} = 2 \ \otimes^1 \aleph_\alpha$$

As can be seen later, these notations greatly facilitate the derivation of the Generalized Continuum Hypothesis.

3. AUGMENTED ZERMELO-FRAENKEL THEORY

To the axioms of ZF theory, we add an axiom [3] that we call the *Axiom of Monotonicity* (AM). Using the operators defined earlier, the axiom can be stated as follows.

Axiom of Monotonicity. *If* $m_1 \le m_2$, $k_1 \le k_2$, *and* $n_1 \le n_2$, *then* $m_1 \otimes^{k_1} n_1 \le m_2 \otimes^{k_2} n_2$.

A direct consequence of the axiom of monotonicity is that, for finite $m > 1$ and $k > 0$,

$$2^{\aleph_\alpha} = 2 \ \otimes^1 \aleph_\alpha \le m \otimes^k \aleph_\alpha \le \aleph_\alpha \otimes^{\aleph_0} \aleph_\alpha = \aleph_{\alpha+1}.$$

When this is combined with Cantor's result that $\aleph_{\alpha+1} \le 2^{\aleph_\alpha}$, we get the following theorem.

Continuum Theorem. $m \otimes^k \aleph_\alpha = \aleph_{\alpha+1}$ *for finite* $m > 1, k > 0$.

If we put $m = 2$ and $k = 1$, we get

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}$$

which is the Generalized Continuum Hypothesis [4,5]. Since GCH implies the Axiom of Choice, we have AC also as a theorem of RST.

4. ENHANCED PREDICATE CALCULUS

We extend the underlying logical basis for the ZF theory, the first-order predicate calculus, with three more derivation rules [6] and call it the Enhanced Predicate Calculus. To state these rules, it is necessary to give some definitions first.

1. $F(x)$: We assume that the formulas of RST can be enumerated. The function $F(x)$ gives the x^{th} formula in the list. In the formation of the formulas we assume that more than one complementation at a time is not allowed, since it does not serve any purpose and will merely complicate matters. We will refer to $F(x)$ as the formula stored at address x. In the list, address 0 is reserved for a special formula G given later.

2. $\overline{x}$: The address at which $\overline{F}(x)$ is stored we call $\overline{x}$. Thus $F(\overline{x}) = \overline{F}(x)$. It is easy to see that $\overline{x}$ is a primitive recursive function of x, and $\overline{\overline{x}} = x$. Roughly, a function is recursive if it can be programmed.

3. $P(x, y)$: The primitive recursive predicate (a very long formula) which says that the formula $F(y)$ is a proof of the formula $F(x)$.

4. $D(x)$: An abbreviation for $\exists y \ P(x, y)$ which says that $F(x)$ can be derived. It is not a recursive predicate.

5. $F(x) \Rightarrow F(y)$: The same as $\overline{D}(x) + D(y)$. When the context makes it clear, we will use $+$ instead of $\vee$, as is common. Similarly we omit $\wedge$ whenever the omission is obvious.

6. $F(0)$: The formula $\sim \exists y P(0, y)$ is stored at address 0. $F(0)$ says that $F(0)$ cannot be derived. We will use the symbol G for $\sim \exists y P(0, y)$ and for uniformity $F(g)$ for $F(0)$. Note that G can also be written as $\overline{D}(g)$ and $\overline{G}$ as $D(g)$. Observe that keeping the formula $\sim \exists y P(0, y)$ at address 0, in no way affects the recursive nature of $F(x)$.

7. $F(c)$: The formula $\sim \exists x\ D(x)D(\overline{x})$ has to appear somewhere in our list, we call that address, c. We will use the symbol C for $\sim \exists x\ D(x)D(\overline{x})$. C says that it is impossible to derive both $F(x)$ and $\overline{F}(x)$. C is read as *consistency* and, $\overline{C}$ as *contradiction.*

Now we can state the three additional derivation rules of EPC. The $\top$ used here is a rotated turnstile symbol with the meaning that the following line can be derived from what preceeds. The meaning of $\perp$ should be obvious.

Validity rule: This rule essentially gives a syntactic definition of *truth.*

$D(u)$
$\top$
$F(u)$

Introspection rule: This rule says that if you have a legitimate derivation of $F(u)$ visibly in front of you, you can conclude that $D(u)$ is true.

$\vdots$
$F(u)$
$\top$
$D(u)$

Contradiction rule: This rule says that any formula that leads to a contradiction cannot be derived.

$F(u)$ ∘ assumption
$\vdots$
$\overline{C}$
$\top$
$C \Rightarrow \overline{D}(u)$

It is legitimate to use both the validity rule and the introspection rule under the assumption of the contradiction rule. This rule we may call *no-proof by contradiction.*

Using these derivation rules, we can derive the incompleteness theorems of RST, *without* using any metalanguage.

FIRST INCOMPLETENESS THEOREM. $C \Rightarrow \overline{D}(g)\overline{D}(\overline{g})$

Proof of $C \Rightarrow \overline{D}(g)$

1. G	∘ assumption
2. $D(g)$	∘ introspection rule on line 1
3. $\overline{D}(g)$	∘ definition of G at line 1
4. $\overline{C}$	∘ from lines 2 and 3
$\top$	
$C \Rightarrow \overline{D}(g)$	∘ contradiction rule

Proof of $C \Rightarrow \overline{D}(\overline{g})$

1. $\overline{G}$	∘ assumption
2. $D(g)$	∘ definition of $\overline{G}$ at line 1
3. G	∘ applying validity rule on line 2
4. $\overline{C}$	∘ from lines 1 and 3
$\top$	
$C \Rightarrow \overline{D}(\overline{g})$	∘ contradiction rule

First Incompleteness Theorem immediately follows. □

SECOND INCOMPLETENESS THEOREM. $C \Rightarrow \overline{D}(c)\overline{D}(\overline{c})$

Proof of $C \Rightarrow \overline{D}(c)$

1. C	∘ assumption
2. $C \Rightarrow \overline{D}(g)$	∘ first incompleteness theorem
3. $\overline{D}(g)$	∘ detachment rule on lines 1 and 2
4. G	∘ $\overline{D}(g)$ at line 3 is the definition of G
5. $D(g)$	∘ applying introspection rule on lines 1 and 4
6. $\overline{C}$	∘ from lines 3 and 5
$\top$	
$C \Rightarrow \overline{D}(c)$	∘ contradiction rule

Proof of $C \Rightarrow \overline{D}(\overline{c})$

1. $\overline{C}$	∘ assumption
$\top$	
$C \Rightarrow \overline{D}(\overline{c})$	∘ contradiction rule

Second Incompleteness Theorem immediately follows. □

5. CONCLUSION

If the axiom of monotonicity and the derivation rules introduced do not produce any contradictions, we can divide the statements of RST into four mutually exclusive categories: F is a *theorem*, if a derivation exists for F, but not for $\overline{F}$. F is a *falsehood*, if a derivation exists for $\overline{F}$, but not for F. F is an *introversion*, if a derivation exists for $\overline{F}$ when F is assumed, and a derivation for F exists when $\overline{F}$ is assumed. F is a *profundity*, if a derivation exists for neither F nor $\overline{F}$, and it is not an introversion. Note that according to our definitions, Generalized Continuum Hypothesis and Axiom of Choice are profundities in ZF theory, whereas they are theorems in RST.

At this stage, the question arises whether consistency C can be introduced as an axiom of RST. From the definition of introversion, it should be clear that an introversion cannot be added as an axiom in RST. The following argument shows that C is an introversion.

1. C	∘ new axiom introduced
2. $D(c)$	∘ applying introspection rule on line 1
3. $C \Rightarrow \overline{D}(c)$	∘ second incompleteness theorem
4. $\overline{D}(c)$	∘ detachment rule on lines 1 and 3
$\top$	
$\overline{C}$	∘ from lines 2 and 4

Since any formula can be derived from contradiction $\overline{C}$, including C, the conclusion is that C is an introversion, and hence cannot be introduced as an axiom of RST.

The main problem of mathematics is to classify the entire set of formulas of RST in the four categories mentioned earlier. Assuming that RST contains no contradictions, we can assign values 0 to the profundities, 1 to the theorems, 2 to the falsehoods, and 3 to the introversions of RST. If we do this and ignore the formula $F(0)$, we get a quaternary number in the interval $[0, 1]$ corresponding to the assignment. The digit at the x^{th} position to the right of the quaternary point will decide the category to which $F(x)$ belongs. We may call this number *reality*, since RST encompasses a substantial portion of human knowledge. Since every formula in RST has to be in one of the four categories, the existence of reality cannot be in question. We may call a number *realizable*, if its decimal expansion can be carried out to *any* arbitrary precision, otherwise it is *unrealizable*. One of the outstanding negative achievements of the twentieth century is the recognition that reality is unrealizable.

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