A Graphic Illustration of Rogers-Ramanujan Identities

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Abstract—Rogers-Ramanujan Identities can be visualized by considering directed paths in appropriate graphs. The derivation of generating functions for partition functions is simpler when we make use of graphs.

Keywords—Rogers-Ramanujan Identities; Directed paths.

1. INTRODUCTION

The purpose of this paper is to bring out and explain the close connection between the partitions of a number [1] and the directed paths in a graph. We use Rogers-Ramanujan Identities (RRIdentities) to illustrate this relationship.

The generating functions for the partitions can be derived simply and more naturally when we approach the problem in terms of graphs. In the last part of this paper two differnt forms for the generating functions appearing in the RRIdentities are derived.

2. NOTATIONS AND DEFINITIONS

The notations and terminology here are mostly as in [1] so that we may omit purely algebraic details from our derivations.

S_2: The set of positive integers congruent to either 1 or 4 modulo 5.

$f(S, m, n)$: Number of partitions of n with the stipulation that the partition is restricted to S, a subset of positive integers, and has exactly m parts.

$p(S, n)$: Number of partitions of n with the stipulation that the partition is restricted to S, a subset of positive integers.

$D_2(n)$: Number of partitions of n using any positive integer as a part, with the stipulation that any two parts differ by at least 2.

First RRIdentity: $p(S_2, n) = D_2(n)$.

T_2: The set of positive integers congruent to either 2 or 3 modulo 5.

$E_2(n)$: Number of partitions of n using integers greater than 1 as parts and with the stipulation that any two parts differ by at least 2.

Second RRIdentity: $p(T_2, n) = E_2(n)$.

Cardinality of a path: Number of edges in the path excluding the edges with label 1.

Extent of a path: Number of edges in the path including the edges with the label 1.

Length of a path: Sum of the exponents of s along the path.

Peak of a partition: The largest part in the partition.

I: Unit matrix.

$u_k(m, n)$: Number of partitions of n with the stipulation that any two parts differ by at least 2, peak is exactly k, and the number of parts is exactly m.

$U_k(m, n)$: Number of partitions of n with the stipulation that any two parts differ by at least 2, peak is at most k, and the number of parts is exactly m.

$U(m,n)$: Number of partitions of n with the stipulation that any two parts differ by at least 2, and the number of parts is exactly m.

$V(m,n)$: Number of partitions of n with the stipulation that 1 is not used as a part, any two parts differ by at least 2, and the number of parts is exactly m.

Jacobi's Triple Product Identity:

$$\prod_{n=0}^{\infty}(1-s^{2n+2})(1+rs^{2n+1})(1+r^{-1}s^{2n+1}) = \sum_{n=-\infty}^{\infty} r^n s^{n^2}.$$

The rest of the notations are given in the text as and when they are required.

3. ROGERS-RAMANUJAN IDENTITIES

The adjacency matrix [2] of the infinite graph shown in Figure 1 can be written as

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ \dots \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & \dots \end{array} \\ \begin{pmatrix} rs & 1 & 0 & 0 & \dots \\ 0 & rs^4 & 1 & 0 & \dots \\ 0 & 0 & rs^6 & 1 & \dots \\ 0 & 0 & 0 & rs^9 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \end{array}.$$

Note that the exponents on the diagonal are restricted to S_2. The (1,4) element of the matrix $(I-A)^{-1}$ can easily be calculated as

$$f_{14}(r,s) = \frac{1}{(1-rs)(1-rs^4)(1-rs^6)(1-rs^9)}.$$

If $f_{14}(r,s)$ is expressed as a power series in r and s, the coefficient of $r^m s^n$ gives the number of paths of cardinality m and length n. It is easy to see that the coefficient of $r^m s^n$ can also be interpreted as the number of partitions of n with exactly m parts if the partition is restricted to the set $\{1,4,6,9\}$.

If we consider paths of infinite extent going towards the right starting from node 1, we get

$$F(r,s) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(S_2,m,n) r^m s^n = \frac{1}{\prod_{k=0}^{\infty}(1-rs^{5k+1})(1-rs^{5k+4})}.$$

The adjacency matrix of the infinite graph shown in Figure 2 can be written as

$$B = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \dots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \dots \end{array} \\ \begin{pmatrix} 0 & rs & rs^2 & rs^3 & rs^4 & \dots \\ 0 & 0 & 0 & rs^3 & rs^4 & \dots \\ 0 & 0 & 0 & 0 & rs^4 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \end{array}.$$

It is tedious to write the inverse of $(I - B)$, but if we call the $(0, k)$ element of the inverse as $g_k(r, s)$, we can write the following recurrence relations:

$$
\begin{aligned}
g_0(r, s) &= 1, \\
g_1(r, s) &= rs, \\
&\vdots \\
g_k(r, s) &= rs^k \sum_{h=0}^{k-2} g_h(r, s).
\end{aligned}
$$

An inspection of Figure 2 shows that $g_k(r, s)$ is the generating function of $u_k(m, n)$ and hence we can write

$$G(r, s) = \sum_{k=0}^{\infty} g_k(r, s),$$

as the generating function of $U(m, n)$.

If we divide the partitions specified by $U(m, n)$ into those containing 1 as a part and those which do not, we get the equation [1]

$$U(m, n) = U(m, n - m) + U(m - 1, n - 2m + 1).$$

This in turn implies that

$$G(r, s) = G(rs, s) + rsG(rs^2, s).$$

Rogers and Ramanujan have given the solution [1] to this functional equation as

$$G(r, s) = \frac{\sum_{n=0}^{\infty} (-1)^n r^{2n} s^{\frac{1}{2}n(5n+1)} (1 - r^2 s^{4n+2})}{(1 - s)(1 - s^2) \dots (1 - s^n) \prod_{k=n+1}^{\infty} (1 - rs^k)}.$$

It follows that $G(1, s)$ is the generating function for $D_2(n)$. Proving the First RRIdentity consists in showing that $F(1, s) = G(1, s)$. We start off with $G(1, s)$ and show that it is equal to $F(1, s)$:

$$G(1, s) = \frac{\sum_{n=0}^{\infty} (-1)^n s^{\frac{1}{2}n(5n+1)} (1 - s^{4n+2})}{\prod_{k=1}^{\infty} (1 - s^k)}.$$

A little manipulation of the numerator gives

$$G(1, s) = \frac{\sum_{n=-\infty}^{\infty} (-1)^n s^{\frac{1}{2}n(5n+1)}}{\prod_{k=1}^{\infty} (1 - s^k)}.$$

Rewriting the numerator using Jacobi's Triple Product Identity [1] gives

$$\begin{aligned} G(1,s) &= \frac{\displaystyle\prod_{n=0}^{\infty}(1-s^{5n+2})(1-s^{5n+3})(1-s^{5n+5})}{\displaystyle\prod_{k=1}^{\infty}(1-s^k)}, \\ &= \frac{1}{\displaystyle\prod_{k=0}^{\infty}(1-s^{5n+1})(1-s^{5n+4})}, \\ &= F(1,s). \end{aligned}$$

We have proved the First RRIdentity. The Second RRIdentity can be proved similarly using Figures 3 and 4. Note that the exponents of s in Figure 3 are restricted to T_2.

4. TWO IDENTITIES

Instead of taking the solution $G(r,s)$ as given by Rogers and Ramanujan, we could go in another direction. Consider

$$G_k(r,s) = \sum_{h=0}^{k} g_h(r,s).$$

We can now write

$$\begin{aligned} G_{k+1}(r,s) &= G_k(r,s) + rs^{k+1}\sum_{h=0}^{k-1} g_h(r,s), \\ &= G_k(r,s) + rs^{k+1}G_{k-1}(r,s). \\ \frac{G_{k+1}(r,s)}{G_k(r,s)} &= 1 + \frac{rs^{k+1}}{\dfrac{G_k(r,s)}{G_{k-1}(r,s)}}. \end{aligned}$$

Since

$$G(r,s) = \lim_{k\to\infty} G_k(r,s),$$

we can write $G(r,s)$ as an infinite product. Thus, we get

$$\prod_{k=1}^{\infty}(1 \oplus_1 rs^k) = \frac{\displaystyle\sum_{n=0}^{\infty}(-1)^n r^{2n} s^{\frac{1}{2}n(5n+1)}(1-r^2s^{4n+2})}{(1-s)(1-s^2)\dots(1-s^n)\displaystyle\prod_{k=n+1}^{\infty}(1-rs^k)},$$

where

$$(1 \oplus_1 rs^k) = 1 + \cfrac{rs^k}{1 + \cfrac{rs^{k-1}}{1 + \cfrac{rs^{k-2}}{1 + \cfrac{\cdots}{1 + \cfrac{rs^2}{1+rs}}}}}.$$

If we consider the generating function for $V(m,n)$, we get

$$\prod_{k=2}^{\infty}(1 \oplus_2 rs^k) = \frac{\displaystyle\sum_{n=0}^{\infty}(-1)^n r^{2n} s^{\frac{1}{2}n(5n+3)}(1-rs^{2n+1})}{(1-s)(1-s^2)\dots(1-s^n)\displaystyle\prod_{k=n+1}^{\infty}(1-rs^k)},$$

where

$$(1 \oplus_2 rs^k) = 1 + \cfrac{rs^k}{1+\cfrac{rs^{k-1}}{1+\cfrac{rs^{k-2}}{1+\cfrac{\cdots}{1+\cfrac{rs^3}{1+rs^2}}}}}.$$

5. CONCLUSION

We have shown that graph theory can be a very useful tool in the study of partitions and that the set of paths between any pair of nodes gives the generating function for some partition function.

REFERENCES

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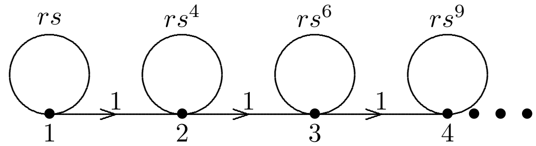

Figure 1

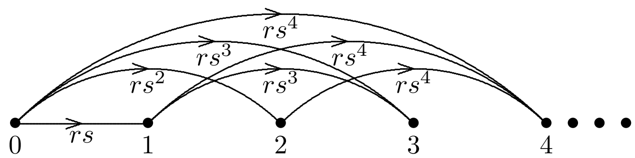

Figure 2

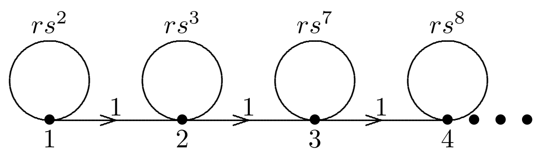

Figure 3

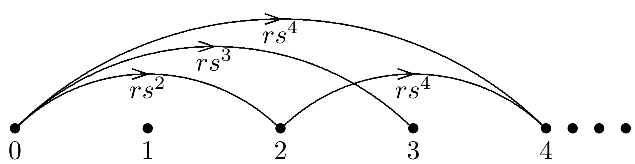

Figure 4