ARROW'S PARADOX AND THE FRACTIONAL VOTING SYSTEM

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ABSTRACT. It is shown that fractional voting system (FVS) can be used to circumvent Arrow's paradox. In the FVS, the input to the voting system is the preference distribution of the voters and not the usual preference order. As a consequence, it turns out that it is possible to associate a unique preference distribution for the society as a whole. An interesting fact is that the same unique distribution results if injustice, as defined here, is minimized.

Keywords—Arrow's paradox, Fractional voting system, Preference distribution.

CONTENTS

1. INTRODUCTION

The 2000 presidential election in the United States clearly demonstrates that our voting systems are far from perfect in taking care of human generated crises. But, even if we assume that we are able to contain these difficulties, a serious fact that that we have to accept is that we will be still left with inherent flaws in our current voting systems. The well-known Arrow's paradox states that, if the input to a voting system is the *preference order* of each voter to the candidates, then there is no way to assign a reasonable preference order that is applicable to the entire society [1, 2, 3]. In what follows, it is shown that the situation changes drastically, if the input to the voting system is changed to the *preference distribution* of each voter. Then it becomes not only possible, but also natural to associate a *unique* preference distribution for the body of voters. The voting system proposed here we will call the *fractional voting system* (FVS).

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In the *voting pattern* matrix K below, a_i is a candidate, b_j is a voter, and k_{ij} is the number of votes given by the voter b_j to the candidate a_i.

$$K = \begin{array}{c c} & \begin{array}{ccccc} b_1 & b_2 & b_3 & \cdots & b_n \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_m \end{array} & \begin{pmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{m1} & k_{m2} & k_{m3} & \cdots & k_{mn} \end{pmatrix} \end{array}$$

In the fractional voting system, each voter b_j has at his disposal not just one vote, but N_j number of votes, and he can distribute these votes to the different candidates in any manner he pleases. Thus we have

$$\sum_{i=1}^{m} k_{ij} = N_j.$$

The total number of votes collected by the candidate a_i is given by

$$\sum_{j=1}^{n} k_{ij} = M_i.$$

The total number of votes cast by the voters is given by

$$\sum_{i=1}^{m} M_i = \sum_{j=1}^{n} N_j = N.$$

After collecting the voting pattern as usual, the candidate who collects the maximum number of votes is declared as the winner by the FVS.

The *preference distribution* matrix of the electorate is defined as,

$$P = \begin{array}{c c} & \begin{array}{ccccc} b_1 & b_2 & b_3 & \cdots & b_n \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_m \end{array} & \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & p_{m3} & \cdots & p_{mn} \end{pmatrix} \end{array}$$

where $p_{ij} = k_{ij}/N$. If we define $q_i = M_i/N$, and $r_j = N_j/N$, then the matrix $[q_i]$ gives the *popularity distribution* of the candidates and the matrix $[r_j]$ gives the *prominence distribution* of the voters within the society. If $q_{ij} = p_{ij}/r_j$, then each column of $Q = [q_{ij}]$ gives the preference distribution of an *individual* voter for the candidates. If $r_{ij} = p_{ij}/q_i$, then each row of $R = [r_{ij}]$ gives the *affinity distribution* of a candidate for the voters.

We can now state the difference between a conventional voting system (CVS) and the FVS. In the CVS, the input to the voting system is the preference *order* of a voter and each voter has a single vote. In the FVS, the input is the preference *distribution* of a voter for the candidates and the prominence distribution of the voters within the society. Arrow has shown that, in the case of CVS, it is impossible for the society to have a reasonable preference order for the candidates. The main purpose of this paper is to show that in the case of FVS, it is possible to have a preference distribution for the candidates satisfying the *unanimity* and *independence* axioms as defined here. Further it is shown that this distribution is unique.

2. DEFINITIONS

Utilizing concepts from information theory [4], it is possible to carry out a thorough analysis of polls conducted under the FVS. The following definitions can be of some use in such an analysis. All logarithms mentioned here are to the base 2. In the sequel, A represents the candidates and B represents voters.

- *Voter hesitance:*
$$H_j(A) = -\sum_{i=1}^{n} q_{ij} \log q_{ij}.$$
- *Voter preference:*
$$I_j(A) = \log m - H_j(A).$$
- *Conditional hesitance:*
$$H(A\overline{B}) = \sum_{j=1}^{n} r_j H_j(A).$$
- *Conditional preference:*
$$I(A\overline{B}) = \log m - H(A\overline{B}).$$
- *Panel homogeneity:*
$$H(A) = -\sum_{i=1}^{m} q_i \log q_i.$$
- *Panel heterogeneity:*
$$I(A) = \log m - H(A).$$
- *Clan uniformity:*
$$H_i(B) = -\sum_{j=1}^{n} r_{ij} \log r_{ij}.$$
- *Clan affinity:*
$$I_i(B) = \log n - H_i(B).$$
- *Conditional uniformity:*
$$H(\overline{A}B) = \sum_{i=1}^{n} q_i H_i(B).$$
- *Conditional affinity:*
$$I(\overline{A}B) = \log n - H(\overline{A}B)$$

∘ *Electorate homogeneity:*

$$H(B) = \sum_{j=1}^{n} r_j \log r_j.$$

∘ *Electorate heterogeneity:*

$$I(B) = \log n - H(B).$$

∘ *Societal homogeneity:*

$$\begin{aligned} H(A+B) &= \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}\log p_{ij} \\ &= H(A) + H(\overline{A}B) \\ &= H(A\overline{B}) + H(B). \end{aligned}$$

∘ *Societal heterogenity:*

$$\begin{aligned} I(A+B) &= \log mn - H(A+B) \\ &= I(A) + I(\overline{A}B) \\ &= I(A\overline{B}) + I(B). \end{aligned}$$

∘ *Election campaign:*

$$\begin{aligned} H(AB) &= H(A) + H(B) - H(A+B) \\ &= H(A) - H(A\overline{B}) \\ &= -H(\overline{A}B) + H(B). \end{aligned}$$

∘ *Election propaganda:*

$$\begin{aligned} I(AB) &= -H(AB) \\ &= I(A) + I(B) - I(A+B) \\ &= I(A) - I(A\overline{B}) \\ &= -I(\overline{A}B) + I(B). \end{aligned}$$

∘ *Popularity of a candidate*:

$$P_i = \log mq_i.$$

∘ a_i is a *popular candidate*, if $P_i \geq 0$.
∘ a_i is an *eminent candidate*, if he is the only popular candidate.
∘ a_i is a *favorite candidate*, if $P_i \geq I(A)$.
∘ a_i is an *outstanding candidate*, if he is the only favorite candidate.
∘ a_i is a *charismatic candidate*, if he collects all the votes without exception.
∘ *Prominence of a voter:*

$$Q_j = \log nr_j.$$

∘ b_j is a *significant voter*, if $Q_j \geq I(B)$.
∘ b_j is a *dominant voter*, if he is the only significant voter.
∘ b_j is a *dictator*, if he has all the votes at his disposal without exception.
∘ An election is a *passive election*, if $I(AB) = 0$.

○ An election is a *dictatorial election*, if $I(AB) = \log mn$.
○ An election is a *positive election*, if there is an eminent candidate.
○ An election is a *definite election*, if there is an outstanding candidate.
○ *Societal preference distribution* $[s_i]$ is the preference chosen by the voting system for the candidates.
○ *Societal injustice to a candidate:*

$$J_i = \log \frac{q_i}{s_i}.$$

○ *Societal injustice to the candidates:*

$$J = \sum_{i=1}^{m} q_i \log \frac{q_i}{s_i}.$$

It is well-known that J can never be negative.

3. PSEPHOLOGY

The following lemmas can be proved from facts well-established in information theory and hence the proofs are omitted.

(1) $P_i = 0$, if and only if, a_i gets exactly the average number of votes. P_i is positive or negative, depending on whether a_i collects above or below the average number of votes. $P_i = \log m$, if and only if, a_i is a charismatic candidate. The maximum value of P_i is $\log m$.
(2) The hierarchy of the candidates is: charismatic, eminent, outstanding, favorite and popular, i.e., each of these classes implies the classes that follow.
(3) In any election, there is at least one favorite candidate.
(4) A positive election is always a definite election.
(5) $I(A) = 0$, if and only if, all the candidates collect equal votes. $I(A) = \log m$, if and only if, there is a charismatic candidate.
(6) $Q_j = 0$, if and only if, b_j has exactly the average number of votes at his disposal. Q_j is positive or negative, depending on whether b_j has above or below the average number of votes at his disposal. $Q_j = \log n$, if and only if, b_j is a dictator. The maximum value of Q_j is $\log n$.
(7) The hierarchy of voters is: dictator, dominant, and significant.
(8) In any election, there is at least one significant voter.
(9) $I(B) = 0$, if and only if, all the voters have equal votes. $I(B) = \log n$, if and only if, there is a dictator.
(10) $I(A\overline{B}) = 0$, if and only if, each individual voter has given equal votes to all the candidates. $I(A\overline{B}) = \log m$, if and only if, each individual voter has all his votes to a single candidate. The maximum value of $I(A\overline{B})$ is $\log m$.
(11) $I(\overline{A}B) = 0$, if and only if, every candidate has received the same number of votes from each voter. $I(\overline{A}B) = \log n$, if and only if, every candidate has got all his votes from a single voter. The maximum possible value of $I(\overline{A}B)$ is $\log n$.

(12) $I(A+B) = 0$, if and only if, each candidate has received the same number of votes from each voter. $I(A+B) = \log mn$, if and only if, there is a charismatic candidate and a dictator. The maximum possible value of $I(A+B)$ is $\log mn$.
(13) Take $\min\{m, n\} = m$. $H(AB) = \log m$, if and only if, each candidate has given all his votes to one candidate and all candidates have collected equal votes.
(14) Take $\min\{m, n\} = n$. $H(AB) = \log n$, if and only if, all voters have equal votes and each candidate has collected all votes from a single voter.
(15) $H(AB) = 0$, if and only if, all the voters have exactly the same preference distribution for the candidates. The maximum possible value of $H(AB)$ is $\min\{m, n\}$.
(16) $J_i = 0$, if and only if, $s_i = q_i$. J_i is positive or negative, depending on whether q_i is greater or less than s_i. Note that J_i can be infinite, both positive and negative.

4. POSSIBILITY THEOREM

Any voting system should have some basic principles by which it ensures a fair election. The fractional voting system conforms to the following two axioms.

Axiom of Unanimity: $q_{11} = q_{12} = \ldots = q_{1n} = q$ implies $s_1 = q$. In words, if each individual voter has the same preference for the candidate a_1, so does the voting system.

Axiom of Independence: $s_i = f(p_{i1}, p_{i2}, \ldots, p_{in})$, i.e., each s_i is the same function of the corresponding row of P. In other words, the voting system does not discriminate between the candidates.

Possibility Theorem. *In the FVS, it is possible to satisfy the axioms of unanimity and independence and to have a societal preference distribution. Further, this distribution is unique.*

Proof. From the given matrix P, construct another matrix P' as given below, where $q = \sum_{j=1}^{n} p_{1j} = q_1$.

$$P' = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ \dfrac{p_{11}(1-q)}{q(m-1)} & \dfrac{p_{11}(1-q)}{q(m-1)} & \cdots & \dfrac{p_{11}(1-q)}{q(m-1)} \\ \dfrac{p_{11}(1-q)}{q(m-1)} & \dfrac{p_{11}(1-q)}{q(m-1)} & \cdots & \dfrac{p_{11}(1-q)}{q(m-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{p_{11}(1-q)}{q(m-1)} & \dfrac{p_{11}(1-q)}{q(m-1)} & \cdots & \dfrac{p_{11}(1-q)}{q(m-1)} \end{pmatrix}$$

It is easy to see that the matrix Q' corresponding to this P' will have

$$q'_{11} = q'_{12} = \ldots = q'_{1n} = q_1 = q.$$

Hence, from the axiom of unanimity we conclude that

$$s_1 = q = q_1 = \sum_{j=1}^{n} p_{1j}.$$

From the axiom of independence we conclude that

$$s_i = q_i = \sum_{j=1}^{n} p_{ij}.$$

□

5. JUSTICE THEOREM

Recall that we defined *societal injustice to the candidates* as

$$J = \sum_{i=1}^{m} q_i \log \frac{q_i}{s_i}.$$

It turns out that minimizing J is equivalent to choosing $[q_i]$ as the societal preference distribution.

Justice Theorem. *J attains the minimum vlaue zero, if and only if, $s_i = q_i$, i.e., the only way to make sure that no injustice is done to the candidates is to choose $[q_i]$ as the societal preference distribution.*

Proof. We use the method of Lagrange multipliers in our proof. Consider

$$U = \sum_{i=1}^{m} q_i \log \frac{q_i}{s_i} + \lambda \sum_{i=1}^{m} s_i.$$

Differentiating U with respect to s_i and equating it zero, gives

$$\begin{aligned}
\frac{\partial U}{\partial s_i} &= -\frac{q_i}{s_i} \log e + \lambda = 0 \\
\frac{q_i}{s_i} &= \frac{\lambda}{\log e} \\
\sum_{i=1}^{m} q_i &= \frac{\lambda}{\log e} \sum_{i=1}^{m} s_i \\
\frac{\lambda}{\log e} &= 1 \\
s_i &= q_i.
\end{aligned}$$

□

We have shown the uniqueness of the societal preference distribution through minimization of societal injustice.

6. IMPLEMENTATION OF FVS

In the fractional voting system, it is useful to consider the last candidate a_n as fictitious and the candidate may be named *anarchy*. All the votes of a voter who protests over the election itself will go to the anarchy candidate. If any voter has utilized some votes, but not all his votes, the unutilized vote will go to anarchy. Any voter who absents himself without protesting against the election will be totally ignored by the FVS. If one or more candidates get disqualified after the voting has taken place, FVS will delete their names from the contest and consider the marginal preference distributions of individual voters with respect to the remaining candidates. If anarchy wins the election, it is an indication of the existence of a substantial group of disgruntled citizens who do not want to participate in the democratic process and the breakdown of democracy. It is interesting to note that FVS caters even to this group of people. The problem faced by the FVS here is the well-known Russel's paradox: What should a true democrat do, when the majority says that they do not want democracy.

7. AN ILLUSTRATIVE EXAMPLE

The matrix below shows a voting pattern in which $N = 32$, $m = 4$, and $n = 8$.

$$[p_{ij}] = \frac{1}{32}\begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 4 & 0 & 8 & 2 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix}$$

$$[q_i] = \frac{1}{32}\begin{pmatrix} 4 \\ 16 \\ 8 \\ 4 \end{pmatrix} \qquad [r_j] = \frac{1}{32}\begin{pmatrix} 4 & 2 & 4 & 4 & 8 & 4 & 2 & 4 \end{pmatrix}$$

Each of the voters b_6 and b_8 used only three of their votes, even though each of them had four votes at their disposal, hence their unused votes have gone to anarchy a_4. Voter b_7 had protested against the election and hence his two votes have been given to anarchy. In this election a_2 gets the highest number of votes, namely 16, and hence gets elected.

○ *Voter preference:*

$$I_1(A) = \frac{1}{2}, \qquad I_2(A) = 2, \qquad I_3(A) = 2, \qquad I_4(A) = 2,$$
$$I_5(A) = 2, \qquad I_6(A) = \frac{1}{2}, \qquad I_7(A) = 2, \qquad I_8(A) = \frac{1}{2}.$$

○ *Conditional preference:*

$$I(A\overline{B}) = \frac{23}{16}.$$

○ *Panel heterogeneity:*

$$I(A) = \frac{1}{4}.$$

○ *Candidate popularity:*

$$P_1 = -1, \qquad P_2 = 1, \qquad P_3 = 0, \qquad P_4 = -1.$$

a_2 is an outstanding candidate and a_3 is a popular candidate.

○ *Clan affinity:*

$$I_1(B) = \frac{3}{2}, \qquad I_2(B) = \frac{5}{4}, \qquad I_3(B) = \frac{5}{4}, \qquad I_4(B) = \frac{3}{4}.$$

○ *Conditional affinity:*

$$I(AB) = \frac{21}{16}.$$

○ *Electorate heterogeneity*

$$I(B) = \frac{1}{8}.$$

○ *Voter prominence:*

$$\begin{array}{llll} Q_1 = 0, & Q_2 = -1, & Q_3 = 0, & Q_4 = 0, \\ Q_5 = 1, & Q_6 = 0, & Q_7 = -1, & Q_8 = 0. \end{array}$$

b_5 is a dominant voter.

○ *Societal heterogeneity:*

$$I(A + B) = \frac{25}{16}.$$

○ *Election campaign:*

$$H(AB) = \frac{19}{16}.$$

8. CONCLUSION

Consider, as an example, the presidential election in India, where the single transferable vote (STV) system is used at present. As a hypothetical case assume that the total value of the votes of all the votes of the members of parliament is 1001. Imagine an election in which 501 votes are in favor of the preference order $a_1a_2a_3a_4a_5$ and the rest 500 votes in favor of the preference order $a_2a_3a_4a_5a_1$. If this situation arises, STV will choose a_1 as the president, which is obviously the wrong choice especially because about half the electorate dislike a_1. Hence, the only inescapable conclusion we can make is that wherever STV is used, it should be discarded in favor of the FVS proposed here. From the possibility and justice theorems given earlier it is clear that anomalous situations cannot occur with FVS. The significant factor here is that the preference distribution used in FVS, gives more freedom to the voter than the preference order used in STV.

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