GENERALIZED CONTINUUM HYPOTHESIS AND THE METHOD OF FUSING

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ABSTRACT. Method of fusing explains the basis for the formulation of the axioms of monotonicity and fusion, the two axioms which define intuitive set theory.

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1. INTRODUCTION

In an earlier paper [3], intuitive set theory was defined as the theory we get when we add the axioms, monotonicity and fusion, to ZF theory. As a matter of record, and also to make this paper reasonably self-contained, we state the two axioms below.

Axiom of Monotonicity. $\aleph_{\alpha+1} = \aleph_\alpha \otimes^{\aleph_0} \aleph_\alpha$, *and* $2^{\aleph_\alpha} = 2 \otimes^1 \aleph_\alpha$. *Further, if* $m_1 \leq m_2$, $k_1 \leq k_2$, *and* $n_1 \leq n_2$, *then* $m_1 \otimes^{k_1} n_1 \leq m_2 \otimes^{k_2} n_2$.

Axiom of Fusion. $(0,1) = \binom{\aleph_\alpha}{\aleph_\alpha} = R \times 2^{\aleph_\alpha}$, *where* $x \times 2^{\aleph_\alpha}$ *is a bonded set.*

Here, *bonded set* is a set from which axiom of choice cannot choose and remove an element, R the class of infinite recursive subsets of positive integers, and $\binom{\aleph_\alpha}{\aleph_\alpha}$ the class of subsets of $\aleph_\alpha$ of cardinality $\aleph_\alpha$. It is known that the cardinality of R is $\aleph_0$.

From these two axioms, two significant theorems follow, one of which is the generalized continuum hypothesis.

Generalized Continuum Hypothesis (GCH). $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$.

Proof. A direct consequence of the axiom of monotonicity is that, for finite $m > 1$ and $k > 0$,

$$2^{\aleph_\alpha} = 2 \ \otimes^1 \aleph_\alpha \leq m \otimes^k \aleph_\alpha \leq \aleph_\alpha \otimes^{\aleph_0} \aleph_\alpha = \aleph_{\alpha+1}.$$

When we combine this with Cantor's result

$$\aleph_{\alpha+1} \leq 2^{\aleph_\alpha},$$

GCH follows. □

Combinatorial Theorem. $\binom{\aleph_\alpha}{\aleph_\alpha} = 2^{\aleph_\alpha}$.

Proof. A direct consequence of the axiom of fusion is that

$$2^{\aleph_\alpha} \leq \binom{\aleph_\alpha}{\aleph_\alpha}.$$

Since, $\binom{\aleph_\alpha}{\aleph_\alpha}$ is a subset of $2^{\aleph_\alpha}$,

$$\binom{\aleph_\alpha}{\aleph_\alpha} \leq 2^{\aleph_\alpha},$$

and the theorem follows. □

If S is a subset of $\aleph_\alpha$ with its complement having cardinality $\aleph_\alpha$, and if we consider only those subsets of $\aleph_\alpha$ of cardinality $\aleph_\alpha$ which contain S, we can show that the cardinality of $\binom{\aleph_\alpha}{\aleph_\alpha}$ will still be $2^{\aleph_\alpha}$. In symbols, we can write this as a theorem.

Conditional Theorem. $\binom{\aleph_\alpha}{\aleph_\alpha}_S = 2^{\aleph_\alpha}$.

As seen above, the generalized continuum hypothesis is an immediate consequence of the axiom of monotonicity. The axiom of fusion says that $(0, 1)$ is a class of bonded sets, called infinitesimals. Clearly these are two powerful axioms, the *method of fusing* explained below gives some indication of the source of their power.

2. INFINITESIMALS

The method of fusing is essentially a method of splitting the unit interval into a set of *infinitesimals*. In the following, we will concentrate on the unit interval, since all the transfinite ordinals can be represented by distinct points in it. It is convenient to talk of points in an interval and for that reason we will use the terms *points* and *elements* interchangeably in the sequel. Note, as an example, that the infinite sequence $.110 * * * * \cdots [\aleph_\alpha]$ can be used to represent the interval $(.75, .875)$, if we accept certain assumptions about the representation:

> The initial binary string, $.110 = .75$, represents the initial point of the interval.
> The length of the binary string, 3 in our case, decides the length of the interval as $2^{-3} = .125$.
> The $\aleph_\alpha$ at the end of the sequence, indicates that the cardinality of the set of $*$s appearing in the sequence is $\aleph_\alpha$.

Every $*$ in the infinite $*$-string can be substituted by a 0 or 1, to create $2^{\aleph_\alpha}$ points in the interval.

This interval is shown as a heavy line in Figure 3 below. The number 3 appearing at the initial point of the interval is obtained by writing the initial binary string 110 in the reverse order as 011 and taking the corresponding decimal number.

Now, consider the sequence $.10101010\cdots[\aleph_0] * * * * \cdots [\aleph_\alpha]$ and the corresponding set $\{1, 3, 5, 7, \cdots [\aleph_0], \cdots [\aleph_\alpha]\}$.

If we can attach a meaning to this sequence, it can be only this: it represents the number $.6666\cdots$ with an *infinitesimal* attached to it, the cardinality of the set of points inside the infinitesimal being $2^{\aleph_\alpha}$.

The nonterminating sequence $.10101010\cdots$ above is an example of an *infinite recursive* sequence. A recursive sequence is a sequence for which there is a clear algorithm for its generation. The number and the subset of positive integers corresponding to a recursive sequence, we will call a computable number and a recursive set, respectively. It is known that there is a one-to-one correspondence between nonterminating recursive sequences and real numbers in the interval $(0, 1)$. Also, the cardinality of the set of nonterminating recursive sequences is $\aleph_0$.

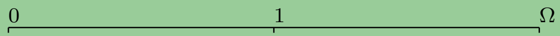

Figure 1

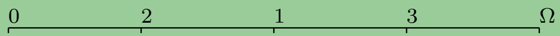

Figure 2

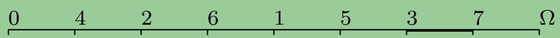

Figure 3

$\vdots$

Figure $\aleph_0$

3. METHOD OF FUSING

With our discussion so far, it is easy to explain the method of fusing and how intuitive set theory visualizes a unit interval. Recall that every real number in the interval $(0, 1)$ can be represented by a sequence of the form

$$.xxxxx\cdots$$

where the xs represent an infinite recursive sequence. Since the cardinality of the set of infinite recursive sequences is $\aleph_0$, we can claim that the unit interval $(0, 1)$ consists of a set of infinitesimals corresponding to the real numbers in it. The cardinality of the set of points in the infinitesimal we can take as $2^{\aleph_\alpha}$, α being any ordinal. If we choose α to be 0, then we get the cardinality of the set of points in $(0, 1)$ as $\aleph_0 2^{\aleph_0} = 2^{\aleph_0}$, and the continuum hypothesis (CH)

$$\aleph_1 = 2^{\aleph_0},$$

as a theorem. When we do not choose any particular value for α, we get the generalized continuum hypothesis

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha},$$

with $2^{\aleph_\alpha}$ as the cardinality of $(0,1)$. A significant difference between ZF theory and IST is that while the former allows access to the elements of an infinitesimal, the latter claims that the elements of an infinitesimal are fused together and cannot be accessed individually, even by the axiom of choice.

4. CONCLUSION

Our purpose in giving the method of fusing was only to explain the motivation for the axioms that define intuitive set theory. Note that the purpose of the method of constructible sets [2] was to prove that CH is consistent with ZF and the purpose of the method of forcing [1] was to prove the consistency of the negation of CH with ZF. Since the method of fusing, like the methods of constructible sets and forcing, deals with the continuum hypothesis, it is not surprising that there is overlap between the concepts used in these methods.

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