DEFINITION OF INTUITIVE SET THEORY

K. K. NAMBIAR

Dedicated to the memory of Professor Narsingh Rao.

Abstract. Two axioms which define intuitive set theory, Axiom of Combinatorial Sets and Axiom of Infinitesimals, are stated. Generalized Continuum Hypothesis is derived from the first axiom, and the infinitesimal is visualized using the latter.

Introduction

In two earlier papers [1, 2], intuitive set theory was defined as the theory we get when we add the two axioms, *monotonicity* and *fusion*, to ZF theory. Here we attempt to replace these two axioms with, Axiom of Combinatorial Sets and Axiom of Infinitesimals, the motivation being that they are simpler to state and possibly easier to accept.

Statement of the Axioms

If k is an ordinal, we will write $\binom{\aleph_\alpha}{k}$ for the cardinality of the set of all subsets of $\aleph_\alpha$ with the same cardinality as k.

Axiom of Combinatorial Sets.

$$\aleph_{\alpha+1} = \binom{\aleph_\alpha}{\aleph_\alpha},$$

where $\binom{\aleph_\alpha}{\aleph_\alpha}$ is the cardinality of the set of all subsets of $\aleph_\alpha$ of cardinality $\aleph_\alpha$.

From the axiom of combinatorial sets, a very significant theorem follows, in fact, the derivation is almost immediate.

Generalized Continuum Hypothesis.

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

Proof.

$$2^{\aleph_\alpha} = \binom{\aleph_\alpha}{0} + \binom{\aleph_\alpha}{1} + \binom{\aleph_\alpha}{2} + \cdots \binom{\aleph_\alpha}{\aleph_0} + \cdots \binom{\aleph_\alpha}{\aleph_\alpha}.$$

Date: October 2, 2001.

2000 *Mathematics Subject Classification.* Primary 03E30; Secondary 03E17.

Note that $\binom{\aleph_\alpha}{1} = \aleph_\alpha$. Since, there are $\aleph_\alpha$ terms in this addition and $\binom{\aleph_\alpha}{k}$ is a monotonically nondecreasing function of k, we can conclude that

$$2^{\aleph_\alpha} = \binom{\aleph_\alpha}{\aleph_\alpha}.$$

Using axiom of combinatorial sets, we get

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

$\square$

We will accept the fact that every number in the interval $(0, 1]$ can be represented *uniquely* by an *infinite* nonterminating binary sequence. For example, the infinite binary sequence

$$.10111111\cdots$$

can be recognized as the representation for the number 3/4 and similarly for other numbers. This in turn implies that an *infinite recursive* subset of positive integers can be used to represent numbers in the interval $(0, 1]$. It is known that the cardinality of the set R of such recursive subsets is $\aleph_0$. Thus, every $r \in R$ represents a real number in the interval $(0, 1]$.

We will write

$$\binom{\aleph_\alpha}{\aleph_\alpha}_r$$

to represent the cardinality of the set of all those subsets of $\aleph_\alpha$ of cardinality $\aleph_\alpha$ which contain r, and also write

$$\left\{ \binom{\aleph_\alpha}{\aleph_\alpha}_r \;\middle|\; r \in R \right\} = \binom{\aleph_\alpha}{\aleph_\alpha}_R.$$

We will define a *bonded sack* as a collection which can appear only on the left side of the binary relation $\in$ and not on the right side. What this means is that a bonded sack has to be considered as an integral unit from which not even the axiom of choice can pick out an element. The definition of a bonded sack is very much similar to that of an urelement. Note that a bonded sack is not a set.

Axiom of Infinitesimals.

$$(0, 1] = \binom{\aleph_\alpha}{\aleph_\alpha}_R,$$

where $\binom{\aleph_\alpha}{\aleph_\alpha}_r$ is a bonded sack.

The concept of a bonded sack is significant in that it puts a limit beyond which the interval $(0, 1]$ cannot be pried any further. The axiom of infinitesimals allows us to visualize the unit interval $(0, 1]$ as a set of bonded sacks, with cardinality $\aleph_0$. Thus, $\binom{\aleph_\alpha}{\aleph_\alpha}_r$ represents an infinitesimal corresponding to the number r in the interval $(0, 1]$.

Conclusion

In view of the fact that we can derive the generalized continuum hypothesis from the axiom of combinatorial sets, it is easy to see that we can use the axiom of combinatorial sets and the axiom of infinitesimals for defining intuitive set theory.

References

1. K. K. Nambiar, *Intuitive Set Theory*, Computers and Mathematics with Applications **39** (1999), no. 1-2, 183–185.
2. K. K. Nambiar, *Real Set Theory*, Computers and Mathematics with Applications **38** (1999), no. 7-8, 167–171.

Formerly, Jawaharlal Nehru University, New Delhi, 110067, India
Current address: 1812 Rockybranch Pass, Marietta, Georgia, 30066-8015
E-mail address: `nambiar@mediaone.net`