THE ESSENCE OF INTUITIVE SET THEORY

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ABSTRACT. Intuitive set theory is defined as the theory we get when we add the axioms, Monotonicity and Fusion, to ZF theory. Axiom of Monotonicity makes the Continuum Hypothesis true, and the Axiom of Fusion splits the unit interval into infinitesimals.

Keywords—Continuum Hypothesis, Axiom of Choice, Infinitesimals.

CONTENTS

1. INTRODUCTION

The primary purpose of this paper is to give a clear definition of intuitive set theory (IST), so that researchers have all the necessary background to investigate the consistency of the two axioms that define IST [1, 2, 4]. Gödel tells us that even though we are not in a position to prove the consistency of a significant theory, we can prove its inconsistency, if it is inconsistent. The secondary purpose of this paper is to explain IST to the novice who has a passing acquaintance with the transfinite cardinals of Cantor.

2. SEQUENCES AND SETS

We will accept the fact that every number in the open interval $(0, 1)$ can be represented *uniquely* by an *infinite* nonterminating binary sequence.

For example, the infinite binary sequence

$$.10101010\cdots$$

can be recognized as the representation for the number $2/3$ and

$$.10111111\cdots$$

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for the number $3/4$. This in turn implies that an infinite subset of positive integers can be used to represent the numbers in the interval $(0, 1)$. Thus we have the set

$$\{1, 3, 5, 7, \cdots\}^+$$

also as a representation for $2/3$. A binary sequence that goes towards the right as above, we will call a *right-sequence* and the corresponding set a *right-set*, to make provision for a *left-sequence* and a *left-set*. It is easy to see that the left sequence

$$\cdots 000010011.$$

and the corresponding left-set

$$^-\{4, 1, 0\}$$

can be used to represent the number 19. In general, any nonnegative integer can be represented by a left-sequence, which eventually ends up in 0s. Two's complement number system allows us to use left-sequences which eventually end up in 1s to represent negative integers. Thus, we have the left-sequence,

$$\cdots 111100101.$$

and the left-set

$$^-\{\cdots 8, 7, 6, 5, 2, 0\}$$

representing the negative number -27. Adding up all these facts, we can claim that a two-way sequence can be used to represent any number on a real line. For example, the sequence,

$$\cdots 000010011.10101010 \cdots$$

and the corresponding two-way set

$$^-\{4, 1, 0 : 1, 3, 5, 7 \cdots\}^+$$

represent the number $19.6666 \cdots$. Similarly, the complement of this sequence,

$$\cdots 111101100.01010101 \cdots$$

and the corresponding two-way set

$$^-\{\cdots 8, 7, 6, 5, 3, 2 : 2, 4, 6, 8 \cdots\}^+$$

represent the negative number $-19.6666 \cdots$. Note the restriction in our definition of a real number: the left sequence must eventually end up in either 1s or 0s. The number system we get when we put no restriction on both the left-sequence and the right-sequence, we will call the *universal number system* (UNS). A universal number, whose left sequence is *not* eventually-periodic, we will call a *supernatural number*. The connection between the transcendental and supernatural numbers is explained next.

3. UNIVERSAL NUMBER SYSTEM

We will first explain why we have excepted eventually-periodic left-sequences from our definition of supernatural numbers. Consider the left-sequence

$$\cdots 101101101001001.$$

with a periodic part $a = 101 = 5$ of length $l_p = 3$ and a nonperiodic part $b = 001001 = 9$ of length $l_n = 6$. We can write the sequence formally as

$$b + \frac{a 2^{l_n}}{1 - 2^{l_p}}$$

which when evaluated gives

$$-\frac{257}{7}.$$

From this we infer that eventually-periodic left-sequences corresponds to negative rational numbers. A similar argument shows that eventually-periodic right-sequences represent positive rational numbers.

We want to show that corresponding to every transcendental number there is a supernatural number. Given a universal number a, the number we get when we flip the two-way infinite string around the binary point, we will write as a^F. Consider the transcendental number

$$\frac{\pi}{4} = \cdots 000.11001000110\cdots$$

and

$$\left(\frac{\pi}{4}\right)^F = \cdots 01100010011.000\cdots$$

which gives the appearance of a number above all natural numbers. It is for this reason, we have called it a supernatural number, but of course, it is no more supernatural than the transcendental number is transcendental. From this example, we infer that corresponding to every transcendental number in the interval $(0, 1)$, there is a supernatural number. More generally, we can say that every irrational number in the interval $(0, 1)$ has a corresponding supernatural number. By definition, an *infinite recursive* subset of positive integers, is an infinite right-set with a clear algorithm for its generation. The corresponding number in the interval $(0, 1)$ is called a computable number. It is known from recursive function theory that the cardinality of the set R of these computable numbers is $\aleph_0$. A number in the interval $(0, 1)$, which is not computable, we will call an *illusive* number. We will have more to say about irrational computable numbers, but before that we want to take a cursory look at the transfinite cardinals of Cantor.

4. TRANSFINITE CARDINALS

Recall that every natural number can be represented by a set as given below.

$$\begin{aligned} \{\} &= 0, \\ \{0\} &= 1, \\ \{0,1\} &= 2, \\ \{0,1,2\} &= 3, \\ &\vdots \end{aligned}$$

The advantage with this method is that we get an elegant way of defining the first transfinite cardinal of Cantor, as

$$\aleph_0 = \{0, 1, 2, 3, \cdots\}.$$

The set of all subsets of a set S is called the *powerset* of S, and written as 2^S. Cantor has shown (diagonal procedure) that the powerset of S will always have greater cardinality than the set S, even when S is an infinite set. An important consequence of this is that we can without end construct bigger and bigger sets,

$$2^{\aleph_0},\ 2^{2^{\aleph_0}},\ 2^{2^{2^{\aleph_0}}},\ \cdots$$

and hence in set theory we cannot have a set which has the highest cardinality. A disappointing consequence is that we cannot have a universal collection as part of set theory and such a collection will always have to be outside the set theory. One-to-one correspondence is the basis on which cardinality is decided, from which it follows that $\aleph_0$ can also be written as

$$\{1, 2, 4, 8, \cdots\} = \{2^0, 2^1, 2^2, 2^3, \cdots\}.$$

As Halmos points out [3], there is confusion in the literature regarding the notation 2^ω, it has been used to represent the above set and also the set $2^{\aleph_0}$, which *in extenso*, can be written as

$$2^{\{0,1,2,3,\cdots\}}.$$

To prevent this confusion, whenever necessary, we will write $2^{\aleph_0}$ as

$$\{< 2^0, 2^1, 2^2, 2^3, \cdots >\}$$

to imply that $2^{\aleph_0}$ is a derived set from

$$\{2^0, 2^1, 2^2, 2^3, \cdots\}.$$

5. INFINITESIMALS

The study of the set of natural numbers gave us the notion of $\aleph_0$. The concept of a powerset makes it clear that there are higher cardinals above $\aleph_0$. Then the question arises, whether there is some other way of generating larger cardinals, other than taking powersets. Cantor has shown that this is possible, and gives us the sequence of transfinite sets of increasing cardinality as

$$\aleph_0, \aleph_1, \aleph_2, \aleph_3, \cdots ,$$

with the understanding that there is no cardinal between $\aleph_\alpha$ and $\aleph_{\alpha+1}$. How exactly this sequence was generated, is an issue that we will take up later, but for the moment we will accept this sequence.

Because of the one-to-one correspondence between the right-sets and the left-sets, we will concentrate our attention on just the right-sets and right-sequences. Note, as an example, that the infinite sequence $.110{*}{*}{*}{*}\cdots$ can be used to represent the interval $(.75, .875)$, if we accept certain assumptions about the representation:

> The initial binary string, $.110 = .75$, represents the initial point of the interval.
> The length of the binary string, 3 in our case, decides the length of the interval as $2^{-3} = .125$.
> Every $*$ in the infinite $*$-string can be substituted by a 0 or 1, to create $2^{\aleph_\alpha}$ points in the interval.

Now, consider the right-sequence

$$.10101010\cdots{*}{*}{*}{*}{*}\cdots$$

and the corresponding right-set

$$\{1, 3, 5, 7, \cdots \aleph_0, \cdots \aleph_\alpha\}^+.$$

If we can attach a meaning to this right-sequence, it can be only this: it represents the number $.6666\cdots$ with an *infinitesimal* attached to it, the cardinality of the set of points inside the infinitesimal being $2^{\aleph_\alpha}$.

6. AXIOM OF FUSION

The upshot of all our discussion so far is the following: The unit interval $(0,1)$ is a set of infinitesimals with cardinality $\aleph_0$, with each infinitesimal representing a computable number. From the method we used in the construction of the infinitesimal, it will not be unreasonable, if we claim that the infinitesimal is an integral unit from which none of its $2^{\aleph_\alpha}$ elements can be removed. A set from which, the axiom of choice (AC) cannot remove an element, we will call a *bonded set* and the elements in it *figments*. If a set contains only bonded sets as its elements, then we will call it a *class of bonded sets* or just *bonded class*. We will use the term *virtual cardinality* to refer to the cardinality of a bonded class. The set of all subsets of $\aleph_\alpha$ of cardinality $\aleph_\alpha$ we will symbolize as $\binom{\aleph_\alpha}{\aleph_\alpha}$. Our saying so, will not, of course, make anything a fact, so we introduce an axiom called *fusion*.

Axiom of Fusion. $(0,1) = \binom{\aleph_\alpha}{\aleph_\alpha} = R \times 2^{\aleph_\alpha}$, *where* $x \times 2^{\aleph_\alpha}$ *is a bonded set.*

The axiom of fusion says that $(0,1)$ is a class of bonded sets, called infinitesimals. Further, the cardinality of each infinitesimal is $2^{\aleph_\alpha}$, and the *virtual cardinality* of $(0,1)$ is $\aleph_0$.

Combinatorial Theorem. $\binom{\aleph_\alpha}{\aleph_\alpha} = 2^{\aleph_\alpha}$.

Proof. A direct consequence of the axiom of fusion is that

$$2^{\aleph_\alpha} \leq \binom{\aleph_\alpha}{\aleph_\alpha}.$$

Since, $\binom{\aleph_\alpha}{\aleph_\alpha}$ is a subset of $2^{\aleph_\alpha}$,

$$\binom{\aleph_\alpha}{\aleph_\alpha} \leq 2^{\aleph_\alpha},$$

and the theorem follows. □

7. EXPLOSIVE OPERATORS

Halmos explains [3] the generation of ω_1, the ordinal corresponding to $\aleph_1$ from ω as given below.

> ... In this way we get successively ω, $\omega 2$, $\omega 3$, $\omega 4$, $\cdots$. An application of the axiom of substitution yields something that follows them all in the same sense in which ω follows the natural numbers; that something is ω^2. After that the whole thing starts over again: ω^2+1, ω^2+2, $\cdots$, $\omega^2+\omega$, $\omega^2+\omega+1$, $\omega^2+\omega+2$, $\cdots$, $\omega^2+\omega 2$, $\omega^2+\omega 2+1$, $\cdots$, $\omega^2+\omega 3$, $\cdots$, $\omega^2+\omega 4$, $\cdots$, $\omega^2 2$, $\cdots$, $\omega^2 3$, $\cdots$, ω^3, $\cdots$, ω^4, $\cdots$, ω^ω, $\cdots$, $\omega^{(\omega^\omega)}$, $\cdots$, $\omega^{(\omega^{(\omega^\omega)})}$, $\cdots$ $\cdots$. The next one after all this is ϵ_0; then come ϵ_0+1, ϵ_0+2, $\cdots$, $\epsilon_0+\omega$, $\cdots$, $\epsilon_0+\omega 2$, $\cdots$, $\epsilon_0+\omega^2$, $\cdots$, $\epsilon_0+\omega^\omega$, $\cdots$, $\epsilon_0 2$, $\cdots$, $\epsilon_0\omega$, $\cdots$, $\epsilon_0\omega^\omega$, $\cdots$, ϵ_0^2, $\cdots$ $\cdots$ $\cdots$.

We want to write the essence of this quotation as terse as possible, for this purpose, we will first define *explosive operators*. For positive integers m and n, we define an infinite sequence of operators as follows.

$$m \otimes^0 n = mn,$$
$$m \otimes^k 1 = m,$$
$$m \otimes^k n = m \otimes^h [m \otimes^h [\cdots [m \otimes^h m]]],$$

where the number of m's in the product is n and $h = k - 1$. It is easy to see that

$$m \otimes^1 n = m^n,$$

$$m \otimes^2 n = m^{m^{\cdot^{\cdot^{\cdot^m}}}},$$

where the number of m's tilting forward is n. We can continue to expand the operators in this fashion further, straining our currently available notations, but it is not relevant for us here. Note that these explosive operators are nothing but the well-known Ackermann functions. We use these operators for symbolizing the transfinite cardinals of Cantor.

8. AXIOM OF MONOTONICITY

Stripped of all verbal explanations, we can write the generation of ω_1 as

$$< 0, 1, 2, \cdots \omega, \cdots \omega^2, \cdots \omega^\omega, \cdots {}^\omega\omega, \cdots, \cdots, \cdots >$$

or in terms of the explosive operators as

$$< 0, 1, 2, \cdots \omega, \cdots \omega \otimes^0 \omega, \cdots \omega \otimes^1 \omega, \cdots \omega \otimes^2 \omega, \cdots, \cdots, \cdots > .$$

Cantor has shown that the cardinality of $\omega \otimes^k \omega$ is $\aleph_0$ for all finite values of k, and hence it is not that we have a sequence here of increasing cardinality. Taking into account this fact, we assert that what the sequence means is that

$$\begin{aligned} \aleph_1 &= \{< 0, 1, 2, \cdots \omega, \cdots \omega \otimes^0 \omega, \cdots \omega \otimes^1 \omega, \cdots \omega \otimes^2 \omega, \cdots >\} \\ &= \aleph_0 \otimes^{\{0,1,2,\cdots\}} \aleph_0 \\ &= \aleph_0 \otimes^{\aleph_0} \aleph_0. \end{aligned}$$

Once this is accepted, a natural extension is that

$$\aleph_{\alpha+1} = \aleph_\alpha \otimes^{\aleph_0} \aleph_\alpha.$$

An inspection of the explosive operators shows that $m \otimes^k n$ is a monotonically increasing function of $m, k,$ and n. Hence it will not be unreasonable to expect $m \otimes^k n$ to remain at least monotonically nondecreasing, when $m, k,$ and n assume tranfinite cardinal values. Our saying all this, will not make it a fact, for that reason we state an axiom called axiom of *monotonicity*. Cantor always wanted his Continuum Hypothesis, $2^{\aleph_0} = \aleph_1$, to be true in his set theory. We now introduce an axiom that accomplishes this, and even more.

Axiom of Monotonicity. $\aleph_{\alpha+1} = \aleph_\alpha \otimes^{\aleph_0} \aleph_\alpha$, *and* $2^{\aleph_\alpha} = 2 \otimes^1 \aleph_\alpha$. *Further, if* $m_1 \le m_2$, $k_1 \le k_2$, *and* $n_1 \le n_2$, *then* $m_1 \otimes^{k_1} n_1 \le m_2 \otimes^{k_2} n_2$.

Continuum Theorem. $\aleph_{\alpha+1} = m \otimes^k \aleph_\alpha$ *for finite* $m > 1, k > 0$.

Proof. A direct consequence of the axiom of monotonicity is that, for finite $m > 1$ and $k > 0$,

$$2^{\aleph_\alpha} = 2 \ \otimes^1 \aleph_\alpha \le m \otimes^k \aleph_\alpha \le \aleph_\alpha \otimes^{\aleph_0} \aleph_\alpha = \aleph_{\alpha+1}.$$

When we combine this with Cantor's result

$$\aleph_{\alpha+1} \le 2^{\aleph_\alpha},$$

the theorem follows. □

Generalized Continuum Hypothesis (GCH). $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$.

Proof. If we put $m = 2$, $k = 1$ in the Continuum Theorem, we get

$$\aleph_{\alpha+1} = 2 \otimes^1 \aleph_\alpha = 2^{\aleph_\alpha},$$

making GCH a theorem. □

Unification Theorem. *All the three sequences*

$$\begin{array}{ccccc} \aleph_0, & \aleph_1, & \aleph_2, & \aleph_3, & \dots \\ \aleph_0, & 2^{\aleph_0}, & 2^{\aleph_1}, & 2^{\aleph_2}, & \dots \\ \aleph_0, & \binom{\aleph_0}{\aleph_0}, & \binom{\aleph_1}{\aleph_1}, & \binom{\aleph_2}{\aleph_2}, & \dots \end{array}$$

represent the same series of cardinals.

Proof. The axiom of monotonicity shows that the first two are the same, and the axiom of fusion shows that the last two are same. □

Axiom of Choice (AC). *Cartesian product of nonempty sets will always be nonempty, even if the product is of an infinite family of sets.*

Proof. GCH implies AC, and we have already proved GCH. □

9. CONCLUSION

A new concept that we have introduced in IST is that of a bonded set containing figments. It is somewhat like the concept of quarks in particle physics, where we know that they are there, but we cannot get one of them isolated. Figments can be very helpful in visualizing the space around us. If we call an infinitesimal with figments in it a *white hole*, we can say that the finite part of our physical space is nothing but a tightly packed set of white holes. Since every irrational number has an infinitesimal attached with it, we can claim that every supernatural number has a *black stretch* attached with it and the physical space beyond the finite part is a *black whole* containing black stretches in it.

IST visualizes an infinite recursive subset of positive integers as a number in the interval $(0, 1)$, with a corresponding infinitesimal. This infinitesimal has in it all the transfinite sets containing the original recursive set.

In measure theory, it has not been possible to date to construct a nonLebesgue measurable set without invoking the axiom of choice. IST does not allow figments to be picked up by the axiom of choice and for that reason, it would not be unreasonable to say that there are no nonLebesgue measurable sets in IST.

If we ignore figments, we can visualize the interval $(0, 1)$ as a set with virtual cardinality $\aleph_0$. As a consequence, the Skolem paradox cannot be a serious problem in IST.

More than anything else, IST tells us to be realistic. It maintains that there are points we cannot touch, and that there are spaces we cannot reach.

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