VISUALIZATION OF INTUITIVE SET THEORY

K. K. NAMBIAR

Abstract. Ackermann functions are used recursively to define the transfinite cardinals of Cantor. Continuum Hypothesis and Axiom of Choice are derived from the definition. An axiom which splits the unit interval into infinitesimals is stated. Using illustrations, the resulting set theory is visualized.

Keywords—Continuum Hypothesis, Axiom of Choice, Visualization.

1. INTRODUCTION AND PRELIMINARIES

The purpose of this paper is to give a visualization of the *Intuitive Set Theory* (IST) described axiomatically in an earlier paper [1]. Some of the basic facts of set theory, we want to state first, but to begin with, let us recognize that for the set theorist, every natural number is a set, for example, the integer 5 is considered as the set

$$\{0, 1, 2, 3, 4\}.$$

Adopting this notation, the entire set of natural numbers can be written as the sets

$$\begin{aligned} \{\} &= 0 \\ \{0\} &= 1 \\ \{0, 1\} &= 2 \\ \{0, 1, 2\} &= 3 \\ &\vdots \end{aligned}$$

The advantage with this method is that we get an elegant way of defining $\aleph_0$, the first transfinite cardinal of Cantor, as

$$\{0, 1, 2, 3, \cdots\},$$

and we will see later that we can define the higher cardinals also in this fashion.

An important characteristic of a set is its size or *cardinality*. Two sets are said to have the same cardinality, if a one-to-one correspondence can be set up between them. The set of all subsets of a set S is called the *powerset* of S. Cantor has shown (diagonal procedure) that the powerset of S will always have greater cardinality than the set S, even when S is an infinite set. An important consequence of this is that we can without end construct bigger and bigger sets,

$$2^{\aleph_0},\ 2^{2^{\aleph_0}},\ 2^{2^{2^{\aleph_0}}},\ \cdots$$

and hence in set theory we cannot have a set which has the highest cardinality. A disappointing consequence is that we cannot have a universal collection as part of set theory and such a collection will always have to be outside the set theory. A

Date: June 25, 2000.

significant set, as important as $2^{\aleph_\alpha}$ is the set of all subsets of $\aleph_\alpha$ of cardinality $\aleph_\alpha$. If we call this set $\binom{\aleph_\alpha}{\aleph_\alpha}$, it can be shown, as done in the sequel that

$$\binom{\aleph_\alpha}{\aleph_\alpha} = 2^{\aleph_\alpha}.$$

It is known in recursive function theory that the cardinality of infinite recursive subsets of natural numbers is $\aleph_0$. We will have occasion to use this fact later.

2. AXIOM OF MONOTONICITY

Consider the following question: If somebody, we will call the person P, travels absolutely straight one meter per second indefinitely, where will P be eventually, and how long will it be, before P reaches there. The obvious unsatisfactory answer is that P will soon get out of sight and we will never know what happened to P after that. This answer is very disturbing to a mathematician and therefore he reformulates the question. In the modified form, P does not travel at uniform velocity, instead, travels the first meter in half a second and from then on, every meter takes only half the time of the previous meter. Now the question is, where will P be in exactly one second. Even though Cantor did not formulate the problem in this fashion, it is indirectly this question that Cantor faced, when he was developing his set theory [2]. Withstanding considerable antagonism from his peers, Cantor tells us that it would not be unreasonable to say that P has reached a distance $\aleph_0$ from the origin. To that, we can add with hindsight that the highest velocity reached by P is

$$2^{\{0,1,2,3,\cdots\}} = 2^{\aleph_0}.$$

From the notation here, it should be clear that $2^{\aleph_0}$ can also be considered as the power set of $\aleph_0$. Having whetted his appetite by the discovery of $\aleph_0$ and $2^{\aleph_0}$, Cantor increased the velocity of P even more than exponential, and went on to discover

$$\aleph_1,\ 2^{\aleph_1},\ \aleph_2,\ 2^{\aleph_2},\ \cdots$$

and finally asked the question: is $2^{\aleph_0} = \aleph_1$?. By this time, the "heaven" reached by Cantor by his travel through the infinities was so enticing for the mathematicians, they raised the more general question: is $2^{\aleph_\alpha} = \aleph_{\alpha+1}$?. This question is considered as one of the most important in mathematics today, IST answers the question in the affirmative.

P's sojourn in Cantor's heaven is what set theory is all about. Unfortunately, we cannot talk about the distance or the speed with which P traveled in any natural language, and hence what we do here is to give the travelogue as simply as possible in terms of suggestive figures. These figures represent an infinite sequence of unit intervals.

To understand the sequence of figures from Fig. P_a to Fig. P_ω, we have to talk about *explosive operators*, which is essentially the same as the well-known Ackermann functions.

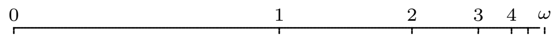

Fig. P_a. Set ω

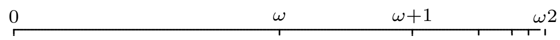

Fig. P_b. Set $\omega 2$

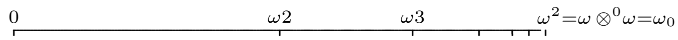

Fig. P_0. Set ω_0

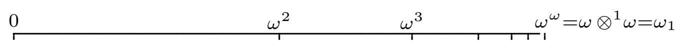

Fig. P_1. Set ω_1

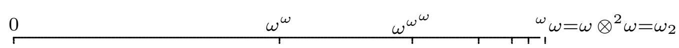

Fig. P_2. Set ω_2

$\vdots$

Fig. P_ω. Set $\aleph_1$

Explosive Operators. For positive integers m and n, we define an infinite sequence of operators as follows.

$$m \otimes^0 n = mn,$$
$$m \otimes^k 1 = m,$$
$$m \otimes^k n = m \otimes^h [m \otimes^h [\cdots [m \otimes^h m]]],$$

where the number of m's in the product is n and $h = k - 1$. It is easy to see that

$$m \otimes^1 n = m^n,$$
$$m \otimes^2 n = m^{m^{\cdot^{\cdot^{\cdot^{m}}}}},$$

where the number of m's tilting forward is n. We can continue to expand the operators in this fashion further, straining our currently available notations, but we will not do so, since it does not serve any purpose here. We have called $\otimes^k$, explosive operators for a good reason. Some idea about the complexity of $\otimes^k$ can be obtained, if one attempts to calculate the number represented by $2 \otimes^3 4$. A detailed evaluation shows that

$$2 \otimes^3 4 = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}},$$

the total number of 2's tilting forward being 65536. We use these operators for symbolizing the transfinite cardinals of Cantor [3].

We remove the restriction on m and n to be positive integers and claim that these operators are meaningful even when m and n take transfinite cardinal values. We go even further and assert that

$$\aleph_{\alpha+1} = \aleph_\alpha \otimes^{\aleph_0} \aleph_\alpha.$$

The reasonableness of this equation can be judged from the fact that $\aleph_1$ can be written in the form

$$\begin{aligned}\aleph_1 &= \{0, 1, 2, \cdots \omega, \cdots \omega^2, \cdots \omega^\omega, \cdots {}^\omega\omega, \cdots, \cdots, \cdots\} \\ &= \{0, 1, 2, \cdots \omega, \cdots \omega \otimes^0 \omega, \cdots \omega \otimes^1 \omega, \cdots \omega \otimes^2 \omega, \cdots \omega \otimes^3 \omega, \cdots, \cdots, \cdots\}.\end{aligned}$$

One more equation we will assert is that the powerset of $\aleph_\alpha$,

$$2^{\aleph_\alpha} = 2 \otimes^1 \aleph_\alpha.$$

It is easy to explain the sojourn of P in terms of the operators $\otimes^k$. In Fig. P_a are marked, the elements of $\omega = \aleph_0$, the natural numbers. The first half of Fig. P_b is obtained by contracting Fig. P_a to half its size, and the second half are marked the elements of $\omega 2$ above ω. The first half of Fig. P_0 is obtained by contracting Fig. P_b to half its size, and the second half are marked the elements of ω^2 above $\omega 2$. By repeating this process indefinitely, we get the infinite sequence of figures, finally ending up with Fig. P_ω. The heavy line of Fig. P_ω indicates that even the smallest of intervals in it anywhere, contains an infinite number of ordinals.

Travel from 0 *to* $\aleph_0$. Fig. P_a explains the travel of P from 0 to $\aleph_0$. To prevent P from going out of sight with exponential speed, we contract the space exponentially every time P takes a step. The distance reached by P at every step is marked in the figure. The distance $\aleph_0$ appears at the end of the unit interval.

Travel from $\aleph_0$ *to* $\aleph_1$. To reach $\aleph_1$, P travels the distance $\aleph_0$ in half-a-second and then steps through the significant ordinals $\omega, \omega 2, \omega^2, \omega^\omega, \ldots$ increasing the speed explosively at every step. To prevent P from going out of sight with explosive speed, we contract the space implosively every time P takes a step. At the end of exactly one second P reaches $\aleph_1$, marked at the end of the unit interval in Fig. P_ω.

Travel from $\aleph_0$ *to* Ω. To reach Ω, the Absolute Infinity, P travels the distance $\aleph_0$ in half-a-second and then steps through the higher cardinals $\aleph_1, \aleph_2, \aleph_3, \ldots$ increasing the speed explosively at every step. To prevent P from going out of sight with explosive speed, we contract the space implosively every time P takes a step. At the end of exactly one second P reaches Ω, which unfortunately is not a cardinal, as shown by Cantor. In a sense, P loses identity and existence, on arrival at Ω.

Cantor always wanted his Continuum Hypothesis, $2^{\aleph_0} = \aleph_1$, to be true in his set theory. We now introduce an axiom to accomplish this, and even more.

Axiom of Monotonicity. *$\aleph_{\alpha+1} = \aleph_\alpha \otimes^{\aleph_0} \aleph_\alpha$, and $2^{\aleph_\alpha} = 2 \otimes^1 \aleph_\alpha$. Further, if $m_1 \le m_2$, $k_1 \le k_2$, and $n_1 \le n_2$, then $m_1 \otimes^{k_1} n_1 \le m_2 \otimes^{k_2} n_2$.*

Continuum Theorem. *$\aleph_{\alpha+1} = m \otimes^k \aleph_\alpha$ for finite $m > 1, k > 0$.*

Proof. A direct consequence of the axiom of monotonicity is that, for finite $m > 1$ and $k > 0$,

$$2^{\aleph_\alpha} = 2 \otimes^1 \aleph_\alpha \le m \otimes^k \aleph_\alpha \le \aleph_\alpha \otimes^{\aleph_0} \aleph_\alpha = \aleph_{\alpha+1}.$$

When we combine this with Cantor's result

$$\aleph_{\alpha+1} \leq 2^{\aleph_\alpha},$$

the theorem follows. □

Generalized Continuum Hypothesis (GCH). $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$.

Proof. If we put $m = 2$, $k = 1$ in the Continuum Theorem, we get

$$\aleph_{\alpha+1} = 2 \ \otimes^1 \aleph_\alpha = 2^{\aleph_\alpha},$$

making GCH a theorem. □

Axiom of Choice (AC). *Given any set S of mutually disjoint nonempty sets, there is a set containing a single member from each element of S.*

Proof. GCH implies AC, and we have already proved GCH. □

3. AXIOM OF FUSION

From the contracted pictures of P's travels all the way to Ω in the mathematical universe, it is clear that the entire set of ordinals can be marked within a unit interval. From this it follows that the interval $[0, 1]$ can be used to represent all the ordinals of Cantor. Now we want to mark the ordinals in $[0, 1]$ rearranged independent of P's travels. In the sequel, the terms *points* and *elements* are used interchangeably.

Note, as an example, that the infinite sequence $.010 * * * * \cdots$ can be used to represent the interval $(.25, .375]$, if we accept certain assumptions about the representation:

> The initial binary string, $.010 = .25$, represents the initial point of the interval.
> The length of the binary string, 3 in our case, decides the length of the interval as $2^{-3} = .125$.
> Every $*$ in the infinite $*$-string can be substituted by a 0 or 1, to create $2^{\aleph_\alpha}$ points in the interval.

We will accept the fact that a *nonterminating* binary sequence, $x = .bbbb\cdots$, or equivalently, an infinite recursive subset of positive integers, can be used *uniquely* to represent a real number in the interval $(0, 1]$.

Consider as an example, the infinite binary sequence $.101010 \cdots ****\cdots$. Using some freedom in terminology, we can say that this represents an *infinitesimal* of length $\aleph_0^{-1}$ located at $\frac{2}{3}$ and the infinitesimal contains $2^{\aleph_\alpha}$ points in it. The length of the infinitesimal is $\aleph_0^{-1}$ because of the fact that the cardinality of R, the class of infinite recursive subsets of positive integers is only $\aleph_0$.

Since an infinite sequence is a precise form of specifying a number, which cannot be improved any further, we can claim that the infinitesimals are sets from which no element can be pried out. More precisely, we can say that, not even the axiom of choice can choose an element from an infinitesimal. We will call a set from which the axiom of choice cannot choose, a *bonded set*, and the elements in it *figments*. The cardinality we get when we ignore figments and take a bonded set as a single element, we will call *virtual cardinality*. The whole idea is made clear by the following axiom [4].

Axiom of Fusion. $(0, 1] = \binom{\aleph_\alpha}{\aleph_\alpha} = R \times 2^{\aleph_\alpha}$, *where $x \times 2^{\aleph_\alpha}$ is a bonded set.*

The axiom of fusion says that $(0,1]$ is a class of bonded sets, called infinitesimals. Further, the significant *combinatorial* part of the power set of $\aleph_\alpha$ consists of $\aleph_0$ bonded sets, each of cardinality $2^{\aleph_\alpha}$. Thus the *virtual cardinality* of $\binom{\aleph_\alpha}{\aleph_\alpha}$ is $\aleph_0$.

We define Intuitive Set Theory as the theory we get when the axioms of monotonicity and fusion are added to Zermelo-Fraenkel set theory (ZF).

Combinatorial Theorem. $\binom{\aleph_\alpha}{\aleph_\alpha} = 2^{\aleph_\alpha}$.

Proof. A direct consequence of the axiom of fusion is that

$$2^{\aleph_\alpha} \le \binom{\aleph_\alpha}{\aleph_\alpha}.$$

Since, $\binom{\aleph_\alpha}{\aleph_\alpha}$ is a subset of $2^{\aleph_\alpha}$,

$$\binom{\aleph_\alpha}{\aleph_\alpha} \le 2^{\aleph_\alpha},$$

and the theorem follows. □

Unification Theorem. *All the three sequences*

$$\begin{array}{lllll} \aleph_0, & \aleph_1, & \aleph_2, & \aleph_3, & \ldots \\ \aleph_0, & 2^{\aleph_0}, & 2^{\aleph_1}, & 2^{\aleph_2}, & \ldots \\ \aleph_0, & \binom{\aleph_0}{\aleph_0}, & \binom{\aleph_1}{\aleph_1}, & \binom{\aleph_2}{\aleph_2}, & \ldots \end{array}$$

represent the same series of cardinals.

Proof. The axiom of monotonicity shows that the first two are the same, and the axiom of fusion shows that the last two are same. □

Fig. 1 should help to build up a mental picture for an infinitesimal. We want to imagine how an infinitesimal part of a unit interval looks like, when magnified $\aleph_0$ times. The heavy line in the figure is, perhaps, as good a representation as any for an infinitesimal, and we can imagine that $\aleph_0$ such infinitesimals constitute a unit interval. The age-old question about a number on the real line is, whether we should consider it as a tiny *iron filing* or as a *steel ball*. According to our view here, it is both. The line (A,B) in the figure is the filing and B is the ball, with the clear understanding that these are only figments of our imagination and can never be palpable.

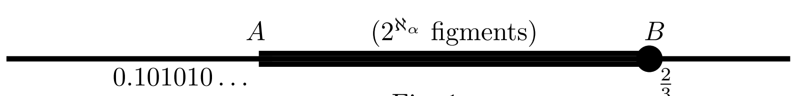

Fig. 1

The binary sequence 0.101010... shown in the figure indicates that the infinitesimal in our visualization corresponds to the number $\frac{2}{3}$ in the unit interval.

If we accept the visualization of an infinitesimal, we can define an *infinitesimal graph*, to represent the whole unit interval. In the Cantorian tradition, we define the infinitesimal graph $G_{\aleph_0}$ as the infinite sequence of graphs shown as Figures $\{G_1, G_3, G_7, \cdots\}$. Note that the graph G_k has k nodes between the nodes 0 and $\aleph_0$, labeled 1 to k. We will take it as axiomatic that the graph $G_{\aleph_0}$ shown in Fig. $G_{\aleph_0}$ has $\aleph_0$ nodes and $\aleph_0$ edges, and also that each infinitesimal consists of an edge and a node. In the graphs, nodes are unconventionally drawn as vertical lines for clarity, and also in deference to Dedekind whose *cut* it represents.

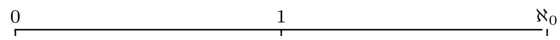

Fig. G_1

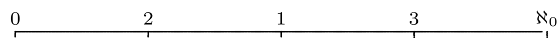

Fig. G_3

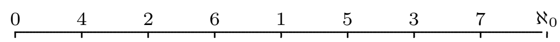

Fig. G_7

$\vdots$

Fig. $G_{\aleph_0}$

Cantor's theorem asserts that every model of Zermelo-Fraenkel set theory has to have cardinality greater than $\aleph_0$. On the other hand, Löwenheim-Skolem theorem (LS) says that there is a model of ZF theory, whose cardinality is $\aleph_0$. These two statements together is called Skolem Paradox.

Intuitive set theory provides a reasonable way to resolve the Skolem Paradox. We merely take the LS theorem as stating that the *virtual* cardinality of a model of IST need not be greater than $\aleph_0$. Clearly, the Upward Löwenheim-Skolem theorem also cannot raise any paradox in IST.

In measure theory, it is known that there are sets which are not Lebesgue measurable, but it has not been possible to date to construct such a set, without invoking the axiom of choice. The usual method is to choose exactly one element from each of the set $x \times 2^{\aleph_\alpha}$ we defined earlier, and show that the set thus created is not Lebesgue measurable. This method is obviously not possible in IST, since $x \times 2^{\aleph_\alpha}$ is a bonded set. Thus, it would not be unreasonable to assert that there are no sets in IST which are not Lebesgue measurable.

4. UNIVERSAL NUMBER SYSTEM

Without losing generality, we will restrict ourselves to binary number system. A number is normally defined as a string $\cdots 000 * * * \cdots * * * . * * * * \cdots$ where the string on the left side of the binary point, has to be all zeroes eventually and the $*$s can take the values either 1 or 0. A slight extension of the 2s complement number system, extensively used in computer science, allows us to conclude that a string of the form $\cdots 111 * * * \cdots * * * . * * * \cdots$, where the left-string ends up in 1s finally, can be used to represent negative numbers.

We define *universal number system* (UNS) as the system we get, when we allow infinite strings on both sides of the binary point, without any restriction. Given a universal number a, the number we get when we flip the two-way infinite string around the binary point, we will write as a^F. Given, a universal number a, the number we get by interchanging the 0s and 1s in it, we will write as $-a$. From our definition, it should be clear that the binary number system is a special case of the universal number system.

As mentioned earlier, every binary string of the form $a = \cdots 000.***\cdots$ represents a number in the interval $[0,1]$. Some examples of as and the corresponding a^Fs are given below.

$$\begin{aligned}
a_0 &= \cdots 000.000 \cdots = 0 & a_0^F &= \cdots 000.000 \cdots = 0\\
a_1 &= \cdots 001.000 \cdots = 1 & a_1^F &= \cdots 000.100 \cdots = \tfrac{1}{2}\\
a_2 &= \cdots 010.000 \cdots = 2 & a_2^F &= \cdots 000.010 \cdots = \tfrac{1}{4}\\
a_3 &= \cdots 011.000 \cdots = 3 & a_3^F &= \cdots 000.110 \cdots = \tfrac{3}{4}\\
a_4 &= \cdots 100.000 \cdots = 4 & a_4^F &= \cdots 000.001 \cdots = \tfrac{1}{8}\\
a_5 &= \cdots 101.000 \cdots = 5 & a_5^F &= \cdots 000.101 \cdots = \tfrac{5}{8}
\end{aligned}$$

The rationale behind the locations and names of the nodes in Fig. $G_{\aleph_0}$ should be clear from the above values.

Consider the transcendental number $\frac{\pi}{4}$ in the interval $[0,1]$.

$$\frac{\pi}{4} = \cdots 000.11001000110 \cdots$$

and

$$\left(\frac{\pi}{4}\right)^F = \cdots 01100010011.000 \cdots$$

which gives the appearance of a number above all natural numbers. For this reason, we will call it a *supernatural number*, of course, it is no more supernatural than the transcendental number is transcendental. From this example, it should be clear that corresponding to every transcendental number in the interval $[0,1]$, there is a supernatural number.

Our discussion shows that the supernatural numbers can be used to represent the entire set of transfinite ordinals. Just as an infinitesimal bonded set is attached with a number in the interval $(0,1)$, an infinite *unreachable stretch* is attached with every supernatural number. Here, the word "stretch" is used as an aid to visualize an infinite set of points spread over an infinite line, but it is to be considered as an element, just as a bonded set is.

5. CONCLUSION

If the two axioms introduced here, do not produce any contradictions in ZF theory, IST produces a simple picture for visualizing all the ordinals of Cantor. The two axioms split the interval $[0,1]$ into $\aleph_0$ bonded sets, each set containing $\aleph_\alpha$ figments. We consider the bonded sets as a single entity with figments in it, which even the axiom of choice cannot access.

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Formerly, Jawaharlal Nehru University, New Delhi, 110067, India
Current address: 1812 Rockybranch Pass, Marietta, Georgia, 30066-8015
E-mail address: nambiar@mediaone.net