LOGSETS AND ZF THEORY

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ABSTRACT. The concept of logsets is introduced into set theory and the consequences explained. It is shown that logsets generate an infinite set of sequences of decreasing numerocity.

CONTENTS

1. INTRODUCTION

The purpose of this paper is to introduce a notion called *logsets* into Zermelo-Fraenkel set theory and explain the consequences. Briefly stated, the operation of getting logsets can be considered as the inverse of the operation of getting powersets. An important fact about logsets is that they produce subsets of $\aleph_0$, with cardinality $\aleph_0$, but with *numerocity* less than $\aleph_0$. An explanation of the notion of numerocity is given below.

2. NUMEROCITY OF A SEQUENCE

Numerocity gives a measure of the frequency with which the elements of a sequence occur within the set of natural number. For example, the numerocity of the sequence $\{1, 3, 5, 7, \ldots\}$ is $\aleph_0/2$, with the implication that the odd numbers occur about half the times within the natural numbers. As another example, the numerocity of the sequence $\{1, 2, 4, 8, \ldots\}$ is $\lg \aleph_0$, with the implication that the frequency decreases logarithmically as we go along the sequence of natural numbers. The prime number theorem tells us that the numerocity of prime numbers is $\aleph_0/\ln \aleph_0$.

The definition of numerocity is as follows. If $s(x)$ is the cardinality of the sequence S truncated at x, and if there is a function $r(x)$, such that

$$\lim_{x\to\infty} \frac{r(x)}{s(x)} = 1$$

then the numerocity of the sequence S is $r(\aleph_0)$.

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3. Log Sequences

Note that ordinals and cardinals are defined by sequences and hence it will be quite meaningful to talk about numerocity $\aleph_\alpha$ of transfinite sequences, in general. However, in the sequel, we will be interested mostly in the sequence of natural numbers

$$\{0, 1, 2, 3, \ldots\}$$

and its subsequences. When the elements of a set are written in the lexical or increasing order, we will call the set a sequence, thus, $\{4, 1, 2\}$ is not a sequence, but $\{1, 2, 4\}$ is.

We want to introduce the notion of a *power sequence* of a sequence. As an example, consider the sequence $S = \{1, 2, 4\}$. The power sequence of S can be written as
$2^S =$
$\{\}$,
$\{1\}, \{2\}, \{4\}$,
$\{1, 2\}, \{1, 4\}, \{2, 4\}$,
$\{1, 2, 4\}$.
Note that all the sets in the power sequence are sequences, since the elements in them are monotonically increasing.

We claim that every power sequence has a *log sequence*, *logset*, or just *lg*, corresponding to it. The log sequence is obtained by collecting the elements of the singleton sets in the power sequence. For our example, the log sequence is

$$\lg 2^S = \{1, 2, 4\}.$$

A generalization of the above example is the following. It is easy to see that for the power sequence
$\{\}$,
$\{1\}, \{2\}, \{4\}, \{8\}, \ldots,$
$\{1, 2\}, \{1, 4\}, \{1, 8\}, \ldots, \{2, 4\}, \{2, 8\}, \{2, 16\}, \ldots,$
$\{1, 2, 4\}, \{1, 2, 8\}, \ldots, \{2, 4, 8\}, \{2, 4, 16\}, \ldots,$
$\ldots$
$\{1, 2, 4, 8, \ldots\}$,
we can write the logset as $\{1, 2, 4, 8, \ldots\}$.
If we add the elements in each set above, we can write the power sequence symbolically as,
0,
$1, 2, 4, 8, \ldots$
$3, 5, 9, \ldots$
$7, 11, \ldots, 14, 22, \ldots$
$\ldots$
$\aleph_0$.
If we rearrange the terms above and write it in a linear fashion, we can claim that the power sequence
$0, 1, 2, 3, 4, \ldots, \aleph_0$
has
$2^0, 2^1, 2^2, 2^3, \ldots$
as its log sequence, and that the numerocity of this log sequence is $\lg \aleph_0$.

Looking at the exponents of the above sequence makes it clear that we can meaningfully take the logset of the logset $\lg \aleph_0$.

4. Definition of Logsets

Cantor has shown that we can produce a larger set from a given set by getting the powerset of the set. Here we want to go in the other direction and produce smaller numerocities from $\aleph_0$, and the way we accomplish this is by introducing a definition for logsets as given below.

Definition of Logsets:

For every transfinite numerocity $\aleph_\alpha$, there exists a transfinite numerocity $\lg \aleph_\alpha$, such that

$$2^{\lg \aleph_\alpha} = \aleph_\alpha.$$

The definition assumes special significance when we consider the numerocity $\lg \aleph_0$. If we symbolize $\lg \aleph_0$ as $\aleph_{-1}$, the definition generates a logset $\aleph_{-2}$, and further an infinitely descending set of numerocities.

5. Conclusion

If the definition of logset does not introduce any contradiction in ZF theory or intuitive set theory [1], we will have the following two-way sequence of numerocities available to us.

$$\ldots, \aleph_{-3}, \aleph_{-2}, \aleph_{-1}, \aleph_0, \aleph_1, \aleph_2, \aleph_3, \ldots$$

From our discussion here, it should be clear that numerocity is nothing but a refinement of the concept of cardinality.

References

1. K. K. Nambiar, *Intuitive Set Theory*, Computers and Mathematics with Applications **39** (2000), no. 1-2, 183–185.

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