TWO OPEN PROBLEMS AND A CONJECTURE IN MATHEMATICAL LOGIC

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ABSTRACT. The first open problem is concerned with introducing three derivation rules in predicate calculus and the second one suggests a solution for the continuum hypothesis. The conjecture says that there will always be profound questions in any significant theory, over and above those suggested by Gödel's incompleteness theorems.

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1. INTRODUCTION

The usual understanding in any axiomatic theory is that there are no contradictions in it and every formula in it ought to be either true or false. If it turns out that a formula cannot be shown to be either true or false, we attempt to remove the flaw by introducing appropriate axioms. This would have worked out very well, but for the fact that Gödel showed that in any significant axiomatic theory, there is always a formula, for which neither itself nor its negation can have a derivation. From this fact, it is claimed that every worthwhile theory is incomplete. Gödel establishes his incompleteness theorems by using metalanguage arguments. To be specific, we will take a theory as *significant*, if it contains Zermelo-Fraenkel set theory.

2. OPEN PROBLEM ONE

A metalanguage has its obvious deficiencies, and our first problem investigates the possibility of proving the incompleteness theorems without using any metalanguage. Following are some notations needed to state the open problem.

$F(x)$: We make use of the known fact that the formulas of an axiomatic theory can be enumerated lexicographically. The function $F(x)$ gives the x^{th} formula in the list. In the formation of the formulas we assume that more than one complementation at a time is not allowed, since it does not serve any purpose. We will refer to $F(x)$ as the formula stored at address x.

Date: January 21, 2002.
2000 *Mathematics Subject Classification.* Primary 03E30; Secondary 03E17.

$\overline{x}$: The address at which $\overline{F}(x)$ is stored we call $\overline{x}$. Thus $F(\overline{x}) = \overline{F}(x)$. It is easy to see that $\overline{x}$ is a primitive recursive function of x, and $\overline{\overline{x}} = x$.

$P(x, y)$: The primitive recursive predicate (a very long formula) which says that the formula $F(y)$ is a proof of the formula $F(x)$.

$D(x)$: An abbreviation for $\exists y\ P(x, y)$ which says that $F(x)$ can be derived. It is not a recursive predicate.

$F(c)$: The formula $\sim \exists x\ D(x)D(\overline{x})$ has to appear somewhere in our list, we call that address, c. We will use the symbol C for $\sim \exists x\ D(x)D(\overline{x})$. C says that it is impossible to derive both $F(x)$ and $\overline{F}(x)$. C is read as *consistency* and, $\overline{C}$ as *contradiction*.

Consider the possibility of adding the following three derivation rules to the *first-order predicate calculus with equality*. The $\top$ used here is a rotated turnstile symbol with the meaning that the following line can be derived from what precedes.

1. Validity rule: This rule essentially gives a syntactic definition of *truth*.

$$\begin{array}{c} D(u) \\ \top \\ F(u) \end{array}$$

2. Introspection rule: This rule says that from a legitimate derivation of $F(u)$, you can conclude that $D(u)$ is true.

$$\begin{array}{c} \vdots \\ F(u) \\ \top \\ D(u) \end{array}$$

3. Contradiction rule: This rule is based on the fact that any formula that leads to a contradiction cannot be derived.

$$\begin{array}{c} F(u) \\ \text{(starting assumption)} \\ \vdots \\ \overline{C} \\ \top \\ C \Rightarrow \overline{D}(u) \end{array}$$

It is legitimate to use both the validity rule and the introspection rule under the assumption of the contradiction rule. This rule we may call *no-proof by contradiction*.

Using these derivation rules, we can derive the incompleteness theorems for any significant theory, *without* using any metalanguage. The derivations are short and straightforward, see [1]. Now, we can state our first problem.

Open Problem 1:
If we use the three derivation rules given above, in a significant theory, will they introduce contradictions in the theory?

Gödel tells us that there is no logical way to assert positively that contradictions will not result, but he also tells us that if contradictions do result, it *can* be proved. It is interesting to note that logic fails us at the most critical juncture.

3. A CONJECTURE

If it is safe to use these derivation rules in an axiomatic theory, then we can divide the formulas in the theory into four mutually exclusive categories: F is a *theorem*, if a derivation exists for F, but not for $\overline{F}$. F is a *falsehood*, if a derivation exists for $\overline{F}$, but not for F. F is an *introversion*, if a derivation exists for $\overline{F}$ when F is assumed, and a derivation for F exists when $\overline{F}$ is assumed. F is a *profundity*, if a derivation exists for neither F nor $\overline{F}$, and it is not an introversion. Note that in ZF theory, according to our classification, generalized continuum hypothesis and axiom of choice are profundities and consistency C is an introversion [2].

An even more serious consequence is that Gödels incompleteness theorems get elevated to a higher level. We will call a theory *profound*, if it contains a profundity. Here is an issue worth considering:

Conjecture:
Every significant theory is profound.

Note that even though continuum hypothesis is a profundity of ZF theory, it is only a theorem of intuitive set theory [2] and hence, we have to look afresh for a meaningful profundity in intuitve set theory.

4. OPEN PROBLEM TWO

It is easy to state the second open problem without any preliminaries, because of the enormous research that has gone into the investigation of the continuum hypothesis in the last century. If k is an ordinal, we will write $\binom{\aleph_\alpha}{k}$ for the cardinality of the set of all subsets of $\aleph_\alpha$ with the same cardinality as k.

Open Problem 2:
If we accept the equation

$$\aleph_{\alpha+1} = \binom{\aleph_\alpha}{\aleph_\alpha},$$

as part of Zermelo-Fraenkel set theory, will it introduce contradictions in the theory?

The importance of the equation above is that if we accept it, the derivation of the generalized continuum hypothesis is almost trivial as outlined below.

$$2^{\aleph_\alpha} = \binom{\aleph_\alpha}{0} + \binom{\aleph_\alpha}{1} + \binom{\aleph_\alpha}{2} + \cdots \binom{\aleph_\alpha}{\aleph_0} + \cdots \binom{\aleph_\alpha}{\aleph_\alpha}.$$

Since, there are $\aleph_\alpha$ terms in this addition and $\binom{\aleph_\alpha}{k}$ is a monotonically nondecreasing function of k, we can conclude that

$$2^{\aleph_\alpha} = \binom{\aleph_\alpha}{\aleph_\alpha}.$$

The generalized continuum hypothesis

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha},$$

immediately follows.

5. CONCLUSION

Note that the categorization of the formulas of a theory remains valid, even if the three derivation rules given earlier turn out to be invalid. All that we have to do is take metamathematical proofs as the basis for the categorization, instead of requiring logical derivations. Thus, the conjecture and the second open problem can be investigated, even when the suggested derivation rules fail.

REFERENCES

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