ELECTRICAL EQUIVALENT OF RIEMANN HYPOTHESIS

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ABSTRACT. Riemann Hypothesis is viewed as a statement about the power dissipated in an electrical network.

CONTENTS

1. INTRODUCTION

Riemann zeta function, an elegantly stated analytic function, has been a fascinating subject of study for generations of mathematicians and it has also produced, the subject of discussion here, an enigma called Riemann hypothesis [1, 2, 3]. The hypothesis simply states that $\zeta_e(s)$, the reciprocal of $\zeta(s)$, has for its abscissa of convergence the line $s = 1/2$. The purpose of this note is to state the hypothesis in terms of the power dissipated in an electrical network.

2. DEFINITIONS AND NOTATIONS

As far as possible, we stick to the traditional notations, but when necessary we make new ones also. Thus, the reader is forewarned that not all the notations here are traditional and standard.

Riemann zeta function $\zeta(s)$: The analytic continuation of the series

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{k=1}^{\infty} \frac{1}{(1 - {p_k}^{-s})}.$$

For uniformity, we write $\zeta(s)$ also as $\zeta_r(s)$.

Riemann delta function $\delta_r(t)$: Defined in terms of Dirac delta function $\delta(t)$ as

$$\delta_r(t) = \sum_{n=1}^{\infty} \delta(t - \log n).$$

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It is easy to show that the Laplace transform of $\delta_r(t)$ is $\zeta_r(s)$, as given below.

$$\begin{aligned}\mathcal{L}[\delta_r(t)] &= \int_0^\infty \sum_{n=1}^{\infty} \delta(t - \log n)e^{-st}\,dt \\ &= \sum_{n=1}^{\infty} e^{-s\log n} \\ &= \sum_{n=1}^{\infty} n^{-s}.\end{aligned}$$

Riemann step function $u_r(t)$: Defined in terms of the right-continuous unit step function $u(t)$ as

$$u_r(t) = \sum_{n=1}^{\infty} u(t - \log n).$$

The stipulation *right-continuous* is not as innocuous as it looks, it is to avoid the usual irritation that arises at discontinuities of functions, due to the classical use of Fourier transforms. What this means is that the Laplace transform captures the entire delta function sitting at the origin, and not just half of it. It is easy to see that the Laplace transform of $u_r(t)$ is given by $\zeta_r(s)/s$.

Inverse zeta function $\zeta_e(s)$: Defined as $1/\zeta_r(s)$.

Euler delta function $\delta_e(t)$: Defined in terms of the Möbius function $\mu(n)$ as

$$\delta_e(t) = \sum_{n=1}^{\infty} \mu(n)\delta(t - \log n).$$

Note that $\mu(n) = 1$, if n can be expressed as a product of an even number of different primes, $\mu(n) = -1$, if n can be expressed as a product of an odd number of different primes, $\mu(1) = 1$, and $\mu(n) = 0$, otherwise.

It is easy to show that the Laplace transform of $\delta_e(t)$ is $\zeta_e(s)$, as given below.

$$\begin{aligned}\mathcal{L}[\delta_e(t)] &= \sum_{n=1}^{\infty} \mu(n)n^{-s} \\ &= \prod_{k=1}^{\infty} (1 - {p_k}^{-s}) \\ &= \zeta_e(s)\end{aligned}$$

Euler step function $u_e(t)$: Defined as

$$u_e(t) = \sum_{n=1}^{\infty} \mu(n)u(t - \log n).$$

Clearly, the Laplace transform of $u_e(t)$ is given by $\zeta_e(s)/s$.

3. Riemann Electrical Hypothesis

In Figure 1 below, what we will call Riemann network, the current generator produces an infinite sequence of current impulses, represented by the Laplace transform $\zeta_e(s)$. To avoid the appearance of impulses in the network and to make the analysis simpler, we take the Thevenin equivalent of the circuit and get the network in Figure 2.

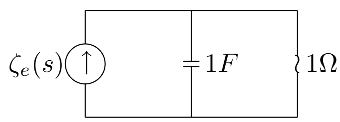

Figure 1.

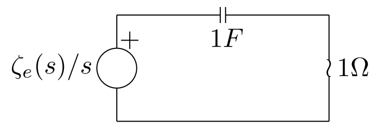

Figure 2.

The Laplace transform $V_r(s)$ of the voltage drop $v_r(t)$ across the resistor and the power $w_r(t)$ dissipated in the resistor, can easily be calculated as below.

$$\begin{aligned} V_r(s) &= \frac{\zeta_e(s)}{s}\frac{s}{s+1} \\ &= \frac{\zeta_e(s)}{s+1}. \end{aligned}$$

This shows that if $\zeta_e(s)$ has for the abscissa of convergence (AC) the line $s = 1/2$, then $V_r(s)$ also has the same AC. From,

$$v_r(t) = \sum_{n=1}^{\infty} \mu(n) u(t - \log n) e^{-(t-\log n)}$$

we can calculate the power dissipated in the resistor as

$$w_r(t) = \Big[\sum_{n=1}^{\infty} \mu(n) u(t - \log n) e^{-(t-\log n)}\Big]^2.$$

If we define $\mathcal{L}[w_r(t)]$ as $W_r(s)$, then it is easy to see that the AC of $W_r(s)$ is $s = 1$, if the AC of $V_r(s)$ is $s = 1/2$. Accepting all these facts, we can restate the Riemann hypothesis as given below.

Riemann Electrical Hypothesis: $W_r(s)$, representing the power dissipated in the Riemann network, has the line $s = 1$ as its abscissa of convergence.

4. Conclusion

Apart from the fact that Riemann zeta function leads us deep into analytic number theory, it is known that it is useful in the study of random Hermitian matrices in quantum mechanics [1]. What we have shown here is that the reciprocal of the Riemann zeta function can be considered as a current source in an electrical network, and that the Riemann hypothesis can be stated in terms of the power dissipated in the network.

References

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3. E. C. Titchmarsh, *The Theory of the Riemann Zeta Function*, Oxford University Press, London, 1986.

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