INFORMATION-THEORETIC EQUIVALENT OF RIEMANN HYPOTHESIS

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ABSTRACT. Riemann Hypothesis is viewed as a statement about the capacity of a communication channel as defined by Shannon.

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1. Introduction

An important concept in information theory is the notion of the *capacity of a communication channel* introduced by Shannon in his classic paper *The Mathematical Theory of Communication* [3]. The purpose of this paper is to show a connection between channel capacity and Riemann zeta functions [1, 2, 4] and to state a conjecture which can possibly be equivalent to the well-known Riemann hypothesis.

2. Definitions and Notations

In the sequel, the sequence of primes $2, 3, 5, \ldots$, is symbolized as

$$p_1, p_2, p_3, \cdots ,$$

it being understood that there are an infinite number of primes. As far as possible, we stick to the traditional notations, but when necessary we make new ones also. Thus, the reader is forewarned that not all the notations here are traditional and standard.

Riemann zeta function $\zeta(s)$*:* The analytic continuation of the series

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{k=1}^{\infty} \frac{1}{(1 - {p_k}^{-s})}.$$

For uniformity, we write $\zeta(s)$ also as $\zeta_r(s)$.

Riemann delta function $\delta_r(t)$*:* Defined in terms of Dirac delta function $\delta(t)$ as

$$\delta_r(t) = \sum_{n=1}^{\infty} \delta(t - \log n).$$

It is easy to show that the Laplace transform of $\delta_r(t)$ is $\zeta_r(s)$, as given below.

$$\begin{aligned}\mathcal{L}[\delta_r(t)] &= \int_0^\infty \sum_{n=1}^\infty \delta(t - \log n) e^{-st}\, dt \\ &= \sum_{n=1}^\infty e^{-s \log n} \\ &= \sum_{n=1}^\infty n^{-s}.\end{aligned}$$

Riemann step function $u_r(t)$: Defined in terms of the right-continuous unit step function $u(t)$ as

$$u_r(t) = \sum_{n=1}^\infty u(t - \log n).$$

Clearly, the Laplace transform of $u_r(t)$ is given by $\zeta_r(s)/s$.

Inverse zeta function $\zeta_e(s)$: Defined as $1/\zeta_r(s)$.

Euler delta function $\delta_e(t)$*:* Defined in terms of the Möbius function $\mu(n)$ as

$$\delta_e(t) = \sum_{n=1}^{\infty} \mu(n)\delta(t - \log n).$$

Note that $\mu(n) = 1$, if n can be expressed as a product of an even number of different primes, $\mu(n) = -1$, if n can be expressed as a product of an odd number of different primes, $\mu(1) = 1$, and $\mu(n) = 0$, otherwise.

It is easy to show that the Laplace transform of $\delta_e(t)$ is $\zeta_e(s)$, as given below.

$$\begin{aligned} \mathcal{L}[\delta_e(t)] &= \sum_{n=1}^{\infty} \mu(n) n^{-s} \\ &= \prod_{k=1}^{\infty} (1 - {p_k}^{-s}) \\ &= \zeta_e(s) \end{aligned}$$

Euler step function $u_e(t)$*:* Defined as

$$u_e(t) = \sum_{n=1}^{\infty} \mu(n) u(t - \log n).$$

Clearly, the Laplace transform of $u_e(t)$ is given by $\zeta_e(s)/s$.

Mertens function $M(n)$*:* Defined as

$$M(n) = \sum_{k=1}^{n} \mu(k).$$

Modified Möbius function $\mu^*(n)$*:* Defined as $\mu^*(1) = 1$ and

$$\mu^*(n) = |M(n)| - |M(n-1)|$$

Shannon zeta function $\zeta_c(s)$*:* Defined as

$$\zeta_c(s) = \sum_{n=1}^{\infty} \mu^*(n) n^{-s}$$

Shannon delta function $\delta_c(t)$*:* Defined as

$$\delta_c(t) = \sum_{n=1}^{\infty} \mu^*(n)\delta(t - \log n).$$

Shannon step function $u_c(t)$*:* Defined as

$$u_c(t) = \sum_{n=1}^{\infty} \mu^*(n)u(t - \log n).$$

Dirichlet series $D(s)$*:* Defined as

$$D(s) = \sum_{n=1}^{\infty} a(n)e^{-\lambda(n)s}$$

where $a(n)$ are complex constants, $\{\lambda(n)\}$ a real monotonically increasing sequence, and s a complex variable.

Shannon series $D_c(s)$*:* Defined as a Dirichlet series in which

$$b(n) = \sum_{k=1}^{n} a(k)$$

are nonnegative integers. Note that we are not insisting that $a(n)$ are nonnegative integers.

Channel series $C(s)$: Defined as a Dirichlet series

$$\sum_{n=1}^{\infty} c(n)e^{-\lambda(n)s}$$

in which $c(n)$ are nonnegative integers. It is easy to see that a channel series can be derived from a Shannon series by defining $c(n)$ as

$$\max\{b(1), b(2), \ldots, b(n)\} - \max\{b(1), b(2), \ldots, b(n-1)\}.$$

From the discussion below, it follows that a channel series can be used to define a communication channel, and hence a Shannon series also.

3. Communication Channels

For the purposes of this paper, we will take the definition of a communication channel as a labelled cyclically connected directed graph with labels from an appropriate alphabet. In his paper [3], Shannon gives the graph for the *telegraph channel* as shown in Figure 1, where a and b respectively represents the closing and opening of a telegraph key for one unit of time.

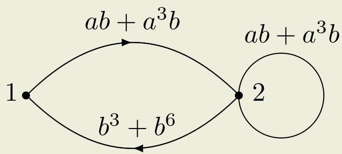

Figure 1.

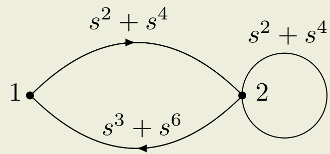

Figure 2.

Figure 2 is obtained from Figure 1 by letting $a = b = s$. The adjacency matrix of this graph can be written as

$$\mathbf{A}(s) = \begin{bmatrix} 0 & s^2 + s^4 \\ s^3 + s^6 & s^2 + s^4 \end{bmatrix},$$

and if we let, $[\mathbf{I} - \mathbf{A}(s)]^{-1} = \mathbf{B}(s)$, we get,

$$\mathbf{B}(s) = \frac{1}{\Delta(s)} \begin{bmatrix} 1 - s^2 - s^4 & s^2 + s^4 \\ s^3 + s^6 & 1 \end{bmatrix}$$

where $\Delta(s) = 1 - s^2 - s^4 - s^5 - s^7 - s^8 - s^{10}$. Thus,

$$b_{11}(s) = \frac{1 - s^2 - s^4}{1 - s^2 - s^4 - s^5 - s^7 - s^8 - s^{10}}.$$

If we consider $b_{11}(s)$ as the generating function of a power series, we get,

$$b_{11}(s) = \sum_{n=0}^{\infty} a(n) s^n,$$

where $a(n)$ are nonnegative integers. $a(n)$ has to be nonnegative because it represents the number of paths in the graph, starting from node 1 and ending in node 1, and containing exactly n symbols.

Shannon defines the channel capacity of the telegraph signal as

$$\begin{aligned} C &= \lim_{n\to\infty} \frac{\log_2 a(n)}{n} \text{ bits}, \\ &= \lim_{n\to\infty} \frac{\log a(n)}{n} \text{ nits}, \end{aligned}$$

where nits (natural units) is defined as $\log_2 e$ bits.

Making use of the fact that

$$\lim_{n\to\infty} \frac{\log_2 a(n)}{n} = -\log_2 s_0,$$

where s_0 is the smallest positive root of the equation

$$\Delta(s) = 1 - s^2 - s^4 - s^5 - s^7 - s^8 - s^{10} = 0,$$

Shannon calculates the capacity of the telegraph channel as

$$-\log_2 0.688278 = 0.538937 \text{ bits}.$$

From the discussion above, it should be clear that

$$b_{11}(e^{-s}) = \frac{1 - e^{-2s} - e^{-4s}}{1 - e^{-2s} - e^{-4s} - e^{-5s} - e^{-7s} - e^{-8s} - e^{-10s}}$$

represents a Shannon series as defined earlier and its abscissa of convergence $s = c_0 + iy$ gives the capacity of the telegraph channel as c_0 nits. In the following, we will consider the Shannon series as the definition of a communication channel and the unit of measure for capacity will be understood as nits.

4. Riemann Metahypothesis

Riemann zeta function is known to be a Dirichlet series, but as the following shows, it is also a Shannon series:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} e^{-s \log n}.$$

If we call the corresponding channel, Riemann channel, we have the capacity of the channel as 1, since the only pole of $\zeta(s)$ is at $s = 1$. We define Shannon channel as the channel corresponding to the Shannon zeta function defined earlier.

Riemann Metahypothesis: The capacity of Shannon channel is 1/2.

5. Conclusion

We call our conjecture metahypothesis, because of our belief that it implies the Riemann hypothesis. The motivation for the conjecture is the hope that no pole of $\zeta_e(s)$ can possibly be lying on the right side of the line $s = 1/2 + iy$. The connection of the metahypothesis to the disproved Mertens conjecture, $|M(x)| < x^{1/2}$, can be recognized if we note that $u_c(\log n) = |M(n)|$.

[1] J. Derbyshire, *Prime Obsession*, Joseph Henry Press, Washington, 2003.
[2] H. M. Edwards, *Riemann's Zeta Function*, Dover Publications, New York, 1974.
[3] C. E. Shannon, *The Mathematical Theory of Communication*, University of Illinois Press, Chicago, 1963.
[4] E. C. Titchmarsh, *The Theory of the Riemann Zeta Function*, Oxford University Press, London, 1986.

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