SHANNON'S COMMUNICATION CHANNELS AND WORD SPACES

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ABSTRACT. A complete analysis of Shannon's telegraph channel is given, making use of matrices with elements from a division ring. A notation is developed for representing the set of signals of a communication channel.

Keywords—Telegraph channel, Labelled graph, Word space.

CONTENTS

1. INTRODUCTION

A diluted version of a vector space, where the underlying algebraic structure is only a division ring and not a field, was called a *word space* in an earlier paper [1], with a definition as below.

$\mathcal{W}$: a commutative group, $\mathbf{w} \in \mathcal{W}$.

$\mathcal{R}$: a division ring, $c \in \mathcal{R}$.

Scalar multiplication: $\mathcal{R} \times \mathcal{W} \to \mathcal{W}$ and $\mathcal{W} \times \mathcal{R} \to \mathcal{W}$.

With these notations, $\mathcal{W}$ is a word space, if it satisfies:

$$\begin{aligned} \mathbf{w}1 &= \mathbf{w} & 1\mathbf{w} &= \mathbf{w} \\ (\mathbf{w}c_1)c_2 &= \mathbf{w}(c_1c_2) & c_1(c_2\mathbf{w}) &= (c_1c_2)\mathbf{w} \\ \mathbf{w}(c_1 + c_2) &= \mathbf{w}c_1 + \mathbf{w}c_2 & (c_1 + c_2)\mathbf{w} &= c_1\mathbf{w} + c_2\mathbf{w} \\ (\mathbf{w}_1 + \mathbf{w}_2)c &= \mathbf{w}_1c + \mathbf{w}_2c & c(\mathbf{w}_1 + \mathbf{w}_2) &= c\mathbf{w}_1 + c\mathbf{w}_2 \end{aligned}$$

The purpose of this paper is to show that the word space can be used to carry out a complete and detailed analysis of Shannon's communication channels [2].

2. TELEGRAPH CHANNEL

Stripped of all technical details, Shannon defines a telegraph channel as a labelled graph as shown in Figure 1, where a and b represents, respectively, the line closing and opening of a telegraph circuit for one unit of time.

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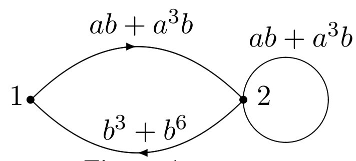

Figure 1.

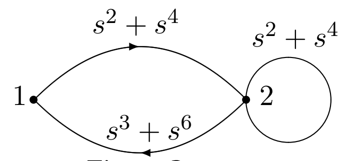

Figure 2.

The adjacency matrix of the graph can be written as

$$\mathbf{A} = \begin{bmatrix} 0 & ab + a^3b \\ b^3 + b^6 & ab + a^3b \end{bmatrix}.$$

If we let,

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

then,

$$\begin{aligned}
b_{11} &= [(ab + a^3b)(ab + a^3b)^\star (b^3 + b^6)]^\star \\
b_{12} &= [(ab + a^3b)(ab + a^3b)^\star (b^3 + b^6)]^\star (ab + a^3b)(ab + a^3b)^\star \\
b_{21} &= [(ab + a^3) + (b^3 + b^6)(ab + a^3b)]^\star (b^3 + b^6) \\
b_{22} &= [(ab + a^3) + (b^3 + b^6)(ab + a^3b)]^\star
\end{aligned}$$

where $\mathbf{I}$ is the unit matrix and $x^\star = 1 + x + x^2 + x^3 \cdots$. The inversion of the matrix $(\mathbf{I} - \mathbf{A})$ can be carried out by slightly generalizing the notion of a regular expression or by using the definition in [1].

We shall call a matrix with elements from a division ring a *page matrix* or just *page* and a row of the page a *line*. It is easy to see from our definition of the word space that the lines of $\mathbf{B}$ defines a linear word space. Also, b_{11} gives a listing of every conceivable telegraph signal possible. The power series we get when we put $a = b = s$ in b_{11}, we shall call $b_{11}(s)$. If we let,

$$b_{11}(s) = \sum_{k=0}^{\infty} n_k s^k,$$

it is easy to see that n_k gives the number of all the possible telegraph signals of length k. Shannon defines the *capacity* of the telegraph channel as

$$C = \lim_{k \to \infty} \frac{\log_2 n_k}{k} \text{ bits per symbol},$$

and shows that the maximum information that can be transmitted through a telegraph channel is C.

3. CHANNEL CAPACITY

Consider Figure 2, which is obtained from Figure 1, by letting $a = b = s$. From our discussion so far, it should be clear that, as far as the investigation of channel capacity is concerned, we could have as well started off with the graph in Figure 2. We shall call the adjacency matrix of this graph

$$\mathbf{A}(s) = \begin{bmatrix} 0 & s^2 + s^4 \\ s^3 + s^6 & s^2 + s^4 \end{bmatrix}.$$

If we let, $[\mathbf{I} - \mathbf{A}(s)]^{-1} = \mathbf{B}(s)$, we get,

$$\mathbf{B}(s) = \frac{1}{\Delta(s)} \begin{bmatrix} 1 - s^2 - s^4 & s^2 + s^4 \\ s^3 + s^6 & 1 \end{bmatrix}$$

where $\Delta(s) = 1 - s^2 - s^4 - s^5 - s^7 - s^8 - s^{10}$. Thus,

$$b_{11}(s) = \frac{1 - s^2 - s^4}{1 - s^2 - s^4 - s^5 - s^7 - s^8 - s^{10}}.$$

Now that we have the generating function for the power series $\sum_{k=0}^{\infty} n_k s^k$, we can make use of a well-known fact in matrix theory:

$$C = \lim_{k \to \infty} \frac{\log_2 n_k}{k} = -\log_2 \mu,$$

where μ is the smallest positive root of the equation $1 - s^2 - s^4 - s^5 - s^7 - s^8 - s^{10} = 0$. Solving this equation gives,

$$\begin{aligned} \mu &= 0.688278, \\ -\log_2 0.688278 &= 0.538937, \end{aligned}$$

which coincides with the value given by Shannon.

4. CONCLUSION

What we have done is to show that there is a word space and a page associated with every communication channel and the capacity of the channel can be calculated from a polynomial matrix $\mathbf{A}(s)$ derived from the page. Since a finite automaton can also be represented by a graph of the kind shown in Figure 1, it is clear that our analysis here is equally applicable to any finite automaton also. Thus, there is a capacity associated with every automaton and the capacity can be calculated from its adjacency matrix. The element b_{11} of the matrix $\mathbf{B}$ gives the signals allowed in the channel.

REFERENCES

1. K. K. Nambiar, *Matrices with Elements from a Division Ring*, Mathematical and Computer Modelling **24** (1996), no. 1, 1–3.
2. C. E. Shannon, *The Mathematical Theory of Communication*, University of Illinois Press, Chicago, 1963.

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