Axiomatic Derivation of the Continuum Hypothesis

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(Received July 1998)

Abstract—After defining Axiom of Monotonicity, it is used along with Zermelo-Fraenkel set theory to derive Cantor's Continuum Hypothesis. Several related theorems are also proved.

Keywords—Transfinite cardinals; Continuum Hypothesis; Derivation.

1. INTRODUCTION

Giving intuitive arguments, an attempt was made in an earlier paper [1] to justify the Continuum Hypothesis (CH), which asserts as a guess that $2^{\aleph_0} = \aleph_1$. The purpose of this paper is to establish that these arguments can be converted into an axiomatic derivation of CH. We show that the derivation can be made from an axiom that we call the *Axiom of Monotonicity* (AM). Thus, if we add AM to Zermelo-Fraenkel set theory (ZF), CH becomes a theorem of ZF.

2. AXIOM OF MONOTONICITY

First, we develop a notation for listing Cantor's transfinite cardinals systematically by defining a sequence of binary operators $\otimes^k$ as given below. Note that m, n, and k can take on only cardinal values with the further restriction that the range of k does not go beyond $\aleph_0$.

$$\begin{aligned} &m \otimes^0 n = m + n, \\ &m \otimes^k n = m \otimes^h [m \otimes^h [\cdots [m \otimes^h m]]], \end{aligned}$$

where the number of m's in the product is n and $h = k - 1$. It is easy to see that

$$\begin{aligned} &m \otimes^1 n = mn, \\ &m \otimes^2 n = m^n, \\ &m \otimes^3 n = m^{m^{\cdot^{\cdot^{m}}}}, \end{aligned}$$

where the number of m's tilting forward is n. For brevity, we will write $m \otimes^3 n$ as ${}^n m$. With these definitions we can write [1] the cardinal $\aleph_1$, in conventional notations as

$$\begin{aligned} \aleph_1 &= \{0, 1, 2, \cdots \omega, \cdots \omega 2, \cdots \omega^2, \cdots \omega^\omega, \cdots {}^\omega\omega, \cdots, \cdots, \cdots\} \\ &= \{0, 1, 2, \cdots \omega, \cdots \omega \otimes^0 \omega, \cdots \omega \otimes^1 \omega, \cdots \omega \otimes^2 \omega, \cdots \omega \otimes^3 \omega, \cdots, \cdots, \cdots\}. \end{aligned}$$

Taking a clue from here we assert that in our notation, a shorter representation is

$$\aleph_1 = \aleph_0 \otimes^{\aleph_0} \aleph_0.$$

Further, we claim that the higher cardinals can be written as

$$\aleph_{\alpha+1} = \aleph_\alpha \otimes^{\aleph_0} \aleph_\alpha,$$

and the power sets as

$$2^{\aleph_\alpha} = 2 \ \otimes^2 \aleph_\alpha.$$

In the derivations that follow, we assume that these notations represent the true meaning of Cantor's transfinite cardinals.

It is easy to see from the definition of $m \otimes^k n$ that it is a monotonically increasing function of m, k, and n, when these take finite values. Keeping this in mind and giving due consideration to the fact that transfinite cardinals are not as natural as natural numbers, we state the following axiom.

Axiom of Monotonicity. *If $m_1 \leq m_2$, $k_1 \leq k_2$, and $n_1 \leq n_2$, then $m_1 \otimes^{k_1} n_1 \leq m_2 \otimes^{k_2} n_2$.*

3. DERIVATION

As a direct consequence of the axiom of monotonicity, we get, for finite $m, k > 1$,

$$2^{\aleph_\alpha} = 2 \ \otimes^2 \aleph_\alpha \leq m \otimes^k \aleph_\alpha \leq \aleph_\alpha \otimes^{\aleph_0} \aleph_\alpha = \aleph_{\alpha+1}.$$

Cantor has shown [2] that $\aleph_{\alpha+1} \leq 2^{\aleph_\alpha}$. Hence, we have the following theorem:

Continuum Theorem. *$m \otimes^k \aleph_\alpha = \aleph_{\alpha+1}$ for finite $m, k > 1$.*

Putting different values for m and k allows us to make several interesting conclusions.

Case 1: Taking $m = 2$, $k = 2$, $\aleph_\alpha = \aleph_0$ gives

$$2^{\aleph_0} = \aleph_1,$$

which is the Continuum Hypothesis.

Case 2: $m = 2$ and $k = 2$ gives

$$2^{\aleph_\alpha} = \aleph_{\alpha+1},$$

which is the Generalized Continuum Hypothesis.

Case 3: $m = 3$, $k = 2$, $\aleph_\alpha = \aleph_0$ gives

$$3^{\aleph_0} = 3 \ \otimes^2 \aleph_0 = \aleph_1.$$

When this is combined with $2^{\aleph_0} = \aleph_1$ we get

$$2^{\aleph_0} = 3^{\aleph_0},$$

which we can state as a theorem:

Radix Theorem. *The cardinality of the set of numbers in the interval $[0, 1]$ does not change when we change the radix from 2 to 3.*

Case 4: $m = 2$ and $k = 3$ gives

$${}^{\aleph_\alpha}2 = \aleph_{\alpha+1},$$

which we may call the Extended Continuum Hypothesis.

4. CONCLUSION

There are two ways to get a larger set from a given set, one way is to add an element to the set and consider it as a new set, and the other way is to generate all the subsets of the given set and consider it as a set. The first method gives the sequence of sets

$$\aleph_0 \otimes^{\aleph_0} \aleph_0 = \aleph_1, \quad \aleph_1 \otimes^{\aleph_0} \aleph_1 = \aleph_2, \quad \aleph_2 \otimes^{\aleph_0} \aleph_2 = \aleph_3, \quad \cdots,$$

and the second method gives the sequence of power sets

$$2 \otimes^2 \aleph_0 = \mathcal{P}_1, \quad 2 \otimes^2 \mathcal{P}_1 = \mathcal{P}_2, \quad 2 \otimes^2 \mathcal{P}_2 = \mathcal{P}_3, \quad \cdots .$$

What we have done is to show that $\aleph_\alpha = \mathcal{P}_\alpha$ and that the two methods generate exactly the same sequence of cardinal numbers. A consequence of this is that it is not necessary to use a binary operator to generate $\aleph_{\alpha+1}$ from $\aleph_\alpha$: if we define $2\otimes^2$ as the unary operator $\prod$, we can write $\aleph_{\alpha+1}$ as $\prod \aleph_\alpha$.

For a graphic visualization of the arguments here, see [1]. Note that there is no possibility of deriving AM from ZF theory, since neither CH nor its negation creates a contradiction [3,4] in ZF.

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