Justification of the Continuum Hypothesis

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Abstract—Distributing the elements of $\aleph_1$ within a unit interval, intuitive arguments are given to justify the Continuum Hypothesis, suggesting that it should be accepted.

Keywords—Transfinite ordinals; Continuum Hypothesis; Justification.

1. INTRODUCTION

In attempting to prove the Continuum Hypothesis (CH), which asserts as a guess that $\aleph_1 = 2^{\aleph_0}$, Cantor [1] has shown that $\aleph_1 \leq 2^{\aleph_0}$. In this paper, we give plausible arguments to demonstrate that $\aleph_1 \geq 2^{\aleph_0}$. If we are successful in this, it is clear that we have justified the CH.

2. DEFINITIONS AND NOTATIONS

One of the difficulties in dealing with the transfinite ordinals is the ad hoc notations used in the theory. For example, the definition of $\aleph_1$ is often given as

$$\aleph_1 = \{0, 1, 2, \cdots \omega, \cdots \omega 2, \cdots \omega^2, \cdots \omega^\omega, \cdots \epsilon_0, \cdots \alpha, \cdots, \cdots, \cdots\}$$

in which, even though the initial part of the sequence can be understood without much difficulty, the symbols after ϵ_0 are not very enlightening. Our first task is to develop a notation which will make the use of the ellipses in the definition of $\aleph_1$ absolutely clear. For this purpose, we define by recursion, an infinite sequence of binary operators as given below. Note that these definitions are applicable only for positive integers and transfinite ordinals:

$$\begin{aligned} &m \otimes^0 n = m + n, \\ &m \otimes^k n = m \otimes^h [m \otimes^h [\cdots [m \otimes^h m]]], \end{aligned}$$

where the number of m's in the product is n and $h = k - 1$. It is easy to see that

$$\begin{aligned} &m \otimes^1 n = mn, \\ &m \otimes^2 n = m^n, \\ &m \otimes^3 n = m^{m^{\cdot^{\cdot^{m}}}}, \end{aligned}$$

where the number of m's tilting forward is n. The interesting fact that comes out is that we have a convenient way of writing the transfinite cardinal $\aleph_1$.

$$\begin{aligned} \aleph_1 &= \{0, 1, 2, \cdots \omega, \cdots \omega \otimes^0 \omega, \cdots \omega \otimes^1 \omega, \cdots \omega \otimes^2 \omega, \cdots, \cdots, \cdots\} \\ &= \{0, 1, 2, \cdots \omega, \cdots \omega_0, \cdots \omega_1, \cdots \omega_2, \cdots, \cdots, \cdots\} \\ &= \omega_{\aleph_0} \\ &= \aleph_0 \otimes^{\aleph_0} \aleph_0 \end{aligned}$$

The next higher cardinal can be written as

$$\begin{aligned}\aleph_2 &= \{0, 1, 2, \cdots \aleph_1, \cdots \aleph_1 \otimes^0 \aleph_1, \cdots \aleph_1 \otimes^1 \aleph_1, \cdots \aleph_1 \otimes^2 \aleph_1, \cdots, \cdots, \cdots\} \\ &= \aleph_1 \otimes^{\aleph_0} \aleph_1,\end{aligned}$$

and similarly other higher cardinals.

3. GRAPHIC VISUALIZATION

Since the Continuum Hypothesis is about the continuity of the real line, our next task is to mark the elements of the transfinite ordinals as points within a unit interval in a systematic fashion. We consider all the significant ordinals less than $\aleph_1$ and use the figures given below as a suggestive medium to justify the inequality $\aleph_1 \geq 2^{\aleph_0}$. Note that the interval shown in each figure is of unit length.

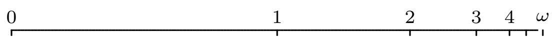

Fig. 1. Set ω

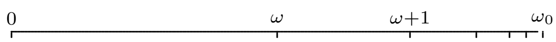

Fig. 2. Set ω_0

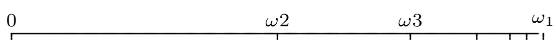

Fig. 3. Set ω_1

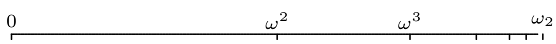

Fig. 4. Set ω_2

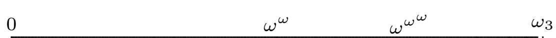

Fig. 5. Set ω_3

⋮

Fig. $\aleph_0$. Set $\aleph_1$

It is useful to take a close look at the markings in each figure. In Fig. 1, the initial point is marked as 0, the midway point is marked as 1, point 2 is marked midway within the rest of the interval, point 3 is marked midway within the rest of the interval and so on. It should be clear then that we can mark all the natural numbers $0, 1, 2, \cdots$ within the interval, in fact, the end point of the interval is still free to be marked. After exhausting all the natural numbers, we mark the free end point of the interval as ω. As the next step we accommodate these entire set of points in the first half of the interval in Fig. 2. This can be done easily by shrinking the unit interval in Fig. 1 to half its size and consider the shrunken interval as the first half of the interval in Fig. 2. Now the second half of the interval is free to be marked. We mark within the interval

$\omega+1, \omega+2, \cdots$ as shown and the extreme point as $\omega 2$. The first half of the interval in Fig. 3 is obtained by shrinking the interval in Fig. 2, and the rest of the interval is marked as shown. The interval between the points $\omega 2$ and $\omega 3$ is obtained by shrinking the interval in Fig. 1. Continuing in this fashion, it is obvious that we can mark the intervals in Fig. 4 and Fig. 5 and, in principle, continue further. The last figure we have is Fig. $\aleph_0$, in which the entire set of elements of $\aleph_1$ is distributed.

From the figures, it should be visibly clear that in the unit interval (0,1] of Fig. $2+k$, the largest segment without points in it, is the initial segment of length $2^{-(2+k)}$ and hence the total number of points in the interval is $\geq 2^{2+k}$. We are keeping the equality sign here deliberately because of the transfinite process we have to go through: If we make the reasonable assumption that this inequality holds good even when k equals $\aleph_0$, we get the required result $\aleph_1 \geq 2^{\aleph_0}$. The validity of CH immediately follows.

With hindsight we can see what must have happened during and after the transfinite process. While in the initial stages the segments remained unequal in length, at the end of the process, all the unequal, unmarked segments became infinitely small and equal to the largest initial segment and $2^{\aleph_0}$ such infinitesimals filled up the entire unit interval. Also, while the cardinality of every ω_k remained constant at $\aleph_0$ for every finite k, the cardinality of $\omega_{\aleph_0}$ jumped to $2^{\aleph_0}$ at the end of the transfinite process. If we are permitted to write the length of the infinitesimals here as $2^{-\aleph_0}$, it is as though that is the smallest length a segment can possibly be shrunk, beyond this, the interval becomes incompressible. All these arguments are, of course, only for those who insist on some visualization of the happenings.

4. CONCLUSION

Instead of starting with ω in Fig. 1, if we start off with $\aleph_k$ and make the corresponding changes in all the figures, we can assert the Generalized Continuum Hypothesis $\aleph_{k+1} = 2^{\aleph_k}$. Since Gödel and Cohen [2,3] have shown that neither the CH nor its negation introduces any contradiction in set theory, it is safe for us to introduce it as an axiom in the theory.

As an aside, we may state that the binary operators we defined earlier are interesting by themselves. $\otimes^0$ is nothing but ordinary addition, $\otimes^1$ is multiplication, $\otimes^2$ is exponentiation, and the higher operators are ones which are not generally used in mathematics. But we could easily imagine what would have happened if these notations were available to Fermat. He would have probably conjectured $(2 \otimes^4 n)+1$ as a prime, instead of his disproved $2^{2^n}+1$. One can make the conjecture even more complicated: There exists a k such that $(2 \otimes^k n)+1$ is a prime for all values of n. A more serious problem would be to investigate whether each of these operators has an analytic continuation. It has been shown in an earlier paper [4] that the Ackermann functions can be written as $m \otimes^k n$. To have a rough estimate of the size of these numbers it should be instructive to compare $2 \otimes^4 4$ with 136×2^{256}, the number of electrons in the universe [5] as given by Eddington.

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