TEACHING
GENERALIZED CONTINUUM HYPOTHESIS

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ABSTRACT. Generalized Continuum Hypothesis is derived from a simple axiom called Axiom of Combinatorial Sets. Instructors are forewarned that the axiom is new and hence should be examined critically before using it.

CONTENTS

1. INTRODUCTION

Cantor has shown that the powerset of a set can never be put into one-to-one correspondence with the original set itself, even when the given set is infinite. What this means is that we can go on taking powerset of powersets to produce larger and larger sets, and thus starting with $\aleph_0$, the set of natural numbers, we can end up with an infinite sequence of infinite sets of ever increasing size. If we use the notation $2^{\aleph_0}$ for the powerset of $\aleph_0$, the natural question that we face is the following: If $\aleph_1$ is the bigger infinity next to $\aleph_0$, is it the same as $2^{\aleph_0}$? In other words, is there an infinity $\aleph_1$, that is larger than $\aleph_0$, but less than $2^{\aleph_0}$? Cantor's guess about the answer to this question is called the Continuum Hypothesis, an issue that has agitated the minds of mathematicians for the whole of last century and continues to do so. If we represent by

$$\aleph_0, \aleph_1, \aleph_2, \aleph_3, \cdots$$

the consecutive tranfinite cardinals of Cantor, the generalized version of the continuum hypothesis can be stated as

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha}.$$

Date: March 29, 2002.

2. PRODUCING $\aleph_1$ FROM $\aleph_0$

Halmos explains [1] the generation of $\aleph_1$ from ω, the ordinal corresponding to $\aleph_0$, as given below.

> ... In this way we get successively ω, $\omega 2$, $\omega 3$, $\omega 4$, $\cdots$. An application of the axiom of substitution yields something that follows them all in the same sense in which ω follows the natural numbers; that something is ω^2. After that the whole thing starts over again: ω^2+1, ω^2+2, $\cdots$, $\omega^2+\omega$, $\omega^2+\omega+1$, $\omega^2+\omega+2$, $\cdots$, $\omega^2+\omega 2$, $\omega^2+\omega 2+1$, $\cdots$, $\omega^2+\omega 3$, $\cdots$, $\omega^2+\omega 4$, $\cdots$, $\omega^2 2$, $\cdots$, $\omega^2 3$, $\cdots$, ω^3, $\cdots$, ω^4, $\cdots$, ω^ω, $\cdots$, $\omega^{(\omega^\omega)}$, $\cdots$, $\omega^{(\omega^{(\omega^\omega)})}$, $\cdots\ \cdots$. The next one after all this is ϵ_0; then come ϵ_0+1, ϵ_0+2, $\cdots$, $\epsilon_0+\omega$, $\cdots$, $\epsilon_0+\omega 2$, $\cdots$, $\epsilon_0+\omega^2$, $\cdots$, $\epsilon_0+\omega^\omega$, $\cdots$, $\epsilon_0 2$, $\cdots$, $\epsilon_0\omega$, $\cdots$, $\epsilon_0\omega^\omega$, $\cdots$, ϵ_0^2, $\cdots\ \cdots\ \cdots$.

While this explanation is very clear and does provide a sophisticated use of ellipses to define $\aleph_1$, it is not of much help in providing a simple solution to the continuum hypothesis. So, we go in another direction.

3. AXIOM OF COMBINATORIAL SETS

If k is an ordinal, we will write $\binom{\aleph_\alpha}{k}$ for the cardinality of the set of all subsets of $\aleph_\alpha$ with the same cardinality as k.

Axiom of Combinatorial Sets:

$$\aleph_{\alpha+1} = \binom{\aleph_\alpha}{\aleph_\alpha}.$$

It turns out that if we accept this axiom, the derivation of the generalized continuum hypothesis becomes very straightforward.

4. DERIVATION

We derive the generalized continuum hypothesis from the axiom of combinatorial sets as below:

$$2^{\aleph_\alpha} = \binom{\aleph_\alpha}{0} + \binom{\aleph_\alpha}{1} + \binom{\aleph_\alpha}{2} + \cdots \binom{\aleph_\alpha}{\aleph_0} + \cdots \binom{\aleph_\alpha}{\aleph_\alpha}.$$

Note that $\binom{\aleph_\alpha}{1} = \aleph_\alpha$. Since, there are $\aleph_\alpha$ terms in this addition and $\binom{\aleph_\alpha}{k}$ is a monotonically nondecreasing function of k, we can conclude that

$$2^{\aleph_\alpha} = \binom{\aleph_\alpha}{\aleph_\alpha}.$$

Using axiom of combinatorial sets, we get

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha},$$

which is the generalized continuum hypothesis.

5. Conclusion

Following are some uncomfortable facts of mathematical logic.

- Gödel tells us that there is no logical method to establish the consistency of any significant axiomatic theory [2]. Here, by *significant* is meant any theory which contains ZF theory. The inconsistency of a theory *can* be proved, if it is inconsistent. The reliability of mathematics depends on the fact that it has worked well for the last two thousand years.
- In any significant axiomatic theory, there is a formula G such that G can be proved when $\overline{G}$ is assumed and $\overline{G}$ can be proved when G is assumed. If we call these formulas *introversions*, the statement of Gödel's incompleteness theorem is that there are introversions in any theory. We will call a formula F a *profundity* if neither F nor $\overline{F}$ can be proved without any assumptions. We will call a theory *profound* if it contains a profundity. Here is a conjecture worth pondering: *Every significant theory is profound*.

It is interesting to note that logic fails us where it is needed most.

6. Epilogue

The Book, an invention of Paul Erdös, the most prolific mathematician of the twentieth century, contains *all* the proofs of mathematical logic listed in the lexical order. This monumental effort is considered the work of God, and it is this book that David Hilbert, the high priest of formalism, once wanted to rewrite with theorems in the lexical order.

Mathematical Logic, under the strict guidance of the court appointed counsel Gödel, uses The Book while taking the oath in the ultimate court of nature: *I solemnly swear that, if I am sane, I will tell the truth, but never the whole truth.*

References

1. P. R. Halmos, *Naive Set theory*, D. Van Nostrand Company, New York, NY, 1960.
2. K. K. Nambiar, *Real Set Theory*, Computers and Mathematics with Applications **38** (1999), no. 7-8, 167–171. http://www.rci.rutgers.edu/~kannan/science/RealSetTheory.pdf.

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