COMPOUND MATRICES AND THREE CELEBRATED THEOREMS

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ABSTRACT. Apart from the regular and adjugate compounds of a matrix, an inverse compound of a matrix is defined. Theorems of Laplace, Binet-Cauchy, and Jacobi are given in terms of these compounds. Compound matrices allow the statements of the theorems to be terse and precise.

Keywords—Inverse compound, Binet-Cauchy theorem, Jacobi's theorem.

CONTENTS

1. INTRODUCTION AND PRELIMINARIES

Study of linear simultaneous equations has led us to consider the set of coefficients in the equations as a single array, and analyze its properties. An important concept in the resulting matrix theory is that of determinants and they play a major role in getting the solutions of simultaneous equations. When we construct matrices with the minors of a determinant as its elements, the resulting matrix is called a *compound matrix* [1]. See below for the exact definition.

The purpose of this paper is to show that three profound theorems of matrix theory, namely, those of Laplace, Binet-Cauchy, and Jacobi, can be stated very elegantly in terms of compound matrices. We give some definitions first.

Matrices I, J, K, and O: As examples, 4×4 matrices of type I, J, K, and O are given below. I has 1s on the main diagonal, J has 1s alternating in sign on the main diagonal, K has 1s on the secondary diagonal (diagonal at right angles to the main diagonal), and O has all elements 0.

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad K = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

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$$O = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$a\binom{i_1 i_2 \cdots i_k}{j_1 j_2 \cdots j_k}$*:* The determinant at the intersection of rows $i_1, i_2, \cdots i_k$ and columns $j_1, j_2, \cdots j_k$ of matrix A. Here, it is assumed that $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.

$A^{(k)}$*:* From a matrix A of order $m \times n$, when the minors of order k are arranged in the lexical order, the resulting $\binom{m}{k} \times \binom{n}{k}$ matrix is called the k^{th} regular compound of A and written as $A^{(k)}$. In symbols, $A^{(k)} = [a\binom{i_1 i_2 \cdots i_k}{j_1 j_2 \cdots j_k}]$.
Thus, $A^{(1)} = [a\binom{i}{j}] = A$.

$A^{\langle k \rangle}$*:* The k^{th} adjugate compound is defined only for square matrices $A(n \times n)$ and it is written as $A^{\langle k \rangle}$. It can be defined in terms of the regular compound as

$$A^{\langle k \rangle} = K\{JA_T J\}^{(n-k)} K.$$

$A^{[k]}$*:* The k^{th} inverse compound is defined only for nonsingular matrices and it is written as $A^{[k]}$. It can be defined in terms of the regular compound as

$$A^{[k]} = \frac{K\{JA_T J\}^{(n-k)} K}{|A|} = \frac{A^{\langle k \rangle}}{|A|}.$$

To firm up our definition of a regular compound matrix, consider the adjacency matrix of a bigraph shown in Figure 1.

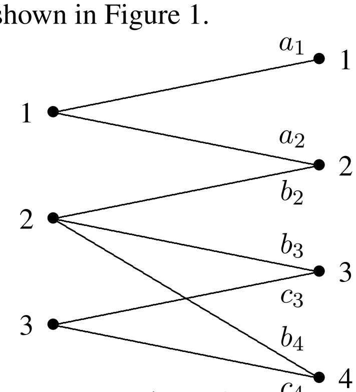

Figure 1.

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} a_1 & a_2 & 0 & 0 \\ 0 & b_2 & b_3 & b_4 \\ 0 & 0 & c_3 & c_4 \end{pmatrix} \end{array} = A^{(1)}$$

$$A^{(2)} = \begin{array}{c} \\ 12 \\ 13 \\ 23 \end{array} \begin{array}{c} \begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \end{array} \\ \begin{pmatrix} a_1b_2 & a_1b_3 & a_1b_4 & a_2b_3 & a_2b_4 & 0 \\ 0 & a_1c_3 & a_1c_4 & a_2c_3 & a_2c_4 & 0 \\ 0 & 0 & 0 & b_2c_3 & b_2c_4 & b_3c_4 - b_4c_3 \end{pmatrix} \end{array}$$

$$A^{(3)} = 123 \begin{array}{c} \begin{array}{cccc} 123 & 124 & 134 & 234 \end{array} \\ \begin{pmatrix} a_1b_2c_3 & a_1b_2c_4 & a_1b_3c_4 - a_1b_4c_3 & a_2b_3c_4 - a_2b_4c_3 \end{pmatrix} \end{array}$$

It easy to see by inspection that the compounds of A lists the *entire* set of *matchings* in the bigraph, a fact that gives some indication of the importance of compound matrices. We mention this only as an aside.

2. THREE THEOREMS

Making use of the definitions given earlier, we can now state the three theorems as below.

Laplace's Theorem.

$$\begin{aligned} A^{(k)}A^{\langle k\rangle} &= |A|\,I, \\ A^{\langle k\rangle}A^{(k)} &= |A|\,I. \end{aligned}$$

Binet-Cauchy Theorem.

$$\begin{aligned} (PQ)^{(k)} &= P^{(k)}Q^{(k)}, \\ (PQ)^{\langle k\rangle} &= Q^{\langle k\rangle}P^{\langle k\rangle}. \end{aligned}$$

Jacobi's Theorem. *If $A^{-1} = B$, then*

$$\begin{aligned} A^{[k]} &= B^{(k)}, \\ A^{(k)} &= B^{[k]}. \end{aligned}$$

Just as a matter of record, we will state these theorems in words, as given in [1].

> *Laplace's theorem.* Take any m rows of $|A|$; no generality is lost by taking the first m rows. From these we may form $n_{(m)}$ minors of order m, where $n_{(m)} = n(n-1)\cdots(n-m+1)/m!$. Multiplying each minor by its cofactor, a signed minor of order $n-m$, we obtain terms of $|A|$.
> *Binet-Cauchy theorem.* The k^{th} compound of a product matrix AB is identically equal to the product of the k^{th} compounds of A and B in that order.
> *Jacobi's theorem.* Any minor of order k in adj A is equal to the complementary signed minor in A', multiplied by $|A|^{k-1}$.

The example given in the next section illustrates the validity of these theorems.

3. VERIFICATION BY EXAMPLE

We will choose a highly structured 4×4 matrix as an example, so that we may verify our computations at every stage. If we take,

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

we get $|A| = 4$,

$$A^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = B,$$

$$A^{(2)} = 2 \begin{pmatrix} -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & -1 & 0 & -1 \end{pmatrix},$$

$$A^{\langle 2 \rangle} = \frac{1}{2} \begin{pmatrix} -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & -1 & 0 & -1 \end{pmatrix},$$

$$A^{[2]} = \frac{1}{8} \begin{pmatrix} -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & -1 & 0 & -1 \end{pmatrix},$$

$$B^{(2)} = \frac{1}{8} \begin{pmatrix} -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & -1 & 0 & -1 \end{pmatrix},$$

$$B^{\langle 2 \rangle} = \frac{1}{2} \begin{pmatrix} -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & -1 & 0 & -1 \end{pmatrix},$$

$$B^{[2]} = 2\begin{pmatrix} -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & -1 & 0 & -1 \end{pmatrix}.$$

Verification of Laplace's theorem. By actual multiplication, we get,

$$A^{(2)}A^{\langle 2\rangle} = 4I \quad = |A|\, I,$$
$$A^{\langle 2\rangle}A^{(2)} = 4I \quad = |A|\, I.$$

Verification of Binet-Cauchy theorem. Since, $AB = I(4\times 4)$, $(AB)^{(2)} = I(6\times 6)$. By actual multiplication, we get, $A^{(2)}B^{(2)} = I(6\times 6)$. Since, $AB = I(4\times 4)$, $(AB)^{\langle 2\rangle} = I(6\times 6)$. By actual multiplication, we get, $B^{\langle 2\rangle}A^{\langle 2\rangle} = I(6\times 6)$. Thus,

$$(AB)^{(2)} = A^{(2)}B^{(2)},$$
$$(AB)^{\langle 2\rangle} = B^{\langle 2\rangle}A^{\langle 2\rangle}.$$

Verification of Jacobi's theorem. It is visibly clear that,

$$A^{[2]} = B^{(2)},$$
$$A^{(2)} = B^{[2]}.$$

Note that the matrix A is orthogonal and symmetric. It can be shown that the compound of an orthogonal matrix is orthogonal, and the compound of a symmetric matrix is symmetric, in fact, this is rather obvious from our discussion.

4. CONCLUSION

There are many facts of matrix theory which become trivial when we think in terms of compounds. For example, a matrix is of rank r, if and only if, $A^{(r)} \neq O$ and $A^{(r+1)} = O$. The coefficients of the characteristic polynomial of A is nothing but the trace of $A^{(k)}$, with an appropriate sign. The rank of AB can never be greater than the lesser of the ranks of the two individual matrices. It was shown in [2] that Hall's theorem can be derived in a few steps if we use compound matrices. However, our intention in this paper was to restrict ourselves to the three celebrated theorems.

REFERENCES

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