TWO AXIOMS TO EXTEND ZERMELO-FRAENKEL THEORY

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ABSTRACT. Axiom of Monotonicity is used along with Zermelo-Fraenkel set theory to derive Generalized Continuum Hypothesis. Axiom of Fusion is used to investigate the cardinality of the set of points in a unit interval.

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1. INTRODUCTION

The purpose of this electronic article is to summarize some of the significant results that appeared in two recent paper publications [1, 2] on the foundations of set theory and give some refinements and extensions of those results. These papers define an axiomatic theory called *Intuitive Set Theory* (IST), in which Generalized Continuum Hypothesis (GCH) and Axiom of Choice (AC) are theorems. A crucial concept in IST is that of a *bonded class* with illusive elements in it, which even the axiom of choice cannot access. The introduction of bonded classes also makes it impossible to produce sets which are not Lebesgue measurable. Further, Skolem Paradox does not arise in IST.

Reasoning about reason is obviously unreasonable, yet that is what we are forced to do when we consider the foundations of mathematics [3]. Accepting this as unavoidable, we add two axioms to Zermelo-Fraenkel (ZF) set theory with the hope that we will not, thereby, introduce contradictions in it. Central to the derivation of GCH in IST is the

Axiom of Monotonicity (AM), which once stated makes the deduction of GCH almost immediate (Section 2). Another axiom in IST called the Axiom of Fusion (AF), converts all sets of cardinality greater than $\aleph_0$ into impregnable bonded classes leaving us with essentially $\aleph_0$ to deal with (Section 3). These two axioms we want to state clearly so that we may examine them critically. What follows is self-contained and does not need any reference to [1, 2].

2. AXIOM OF MONOTONICITY

Here is how Halmos explains [4] the generation of ω_1, the ordinal corresponding to $\aleph_1$ from ω.

> ... In this way we get successively $\omega, \omega 2, \omega 3, \omega 4, \cdots$. An application of the axiom of substitution yields something that follows them all in the same sense in which ω follows the natural numbers; that something is ω^2. After that the whole thing starts over again: $\omega^2+1, \omega^2+2, \cdots, \omega^2+\omega, \omega^2+\omega+1, \omega^2+\omega+2, \cdots, \omega^2+\omega 2, \omega^2+\omega 2+1, \cdots, \omega^2+\omega 3, \cdots, \omega^2+\omega 4, \cdots, \omega^2 2, \cdots, \omega^2 3, \cdots, \omega^3, \cdots, \omega^4, \cdots, \omega^\omega, \cdots, \omega^{(\omega^\omega)}, \cdots, \omega^{(\omega^{(\omega^\omega)})}, \cdots \cdots$. The next one after all this is ϵ_0; then come $\epsilon_0+1, \epsilon_0+2, \cdots, \epsilon_0+\omega, \cdots, \epsilon_0+\omega 2, \cdots, \epsilon_0+\omega^2, \cdots, \epsilon_0+\omega^\omega, \cdots, \epsilon_0 2, \cdots, \epsilon_0\omega, \cdots, \epsilon_0\omega^\omega, \cdots, \epsilon_0^2, \cdots \cdots \cdots$.

This explanation, perhaps one of the best available, is satisfactory if we are interested only in understanding what transfinite numbers are. But, if we want to go beyond and investigate the properties of these numbers, then we have to look for more terse notations. Here is a solution that looks promising.

2.1. Explosive Operators. For positive integers m and n, we define an infinite sequence of operators as follows.

$$m \otimes^0 n = mn,$$
$$m \otimes^k 1 = m,$$
$$m \otimes^k n = m \otimes^h [m \otimes^h [\cdots [m \otimes^h m]]],$$

where the number of m's in the product is n and $h = k - 1$. It is easy to see that

$$m \otimes^1 n = m^n,$$
$$m \otimes^2 n = m^{m^{\cdot^{\cdot^{\cdot^{m}}}}},$$

where the number of m's tilting forward is n. We can continue to expand the operators in this fashion further, straining our currently available notations, but we will not do so, since it does not serve any purpose here. We use these operators for symbolizing the transfinite cardinals of Cantor.

We remove the restriction on m and n to be positive integers and claim that these operators are meaningful even when m and n take transfinite cardinal values. We go even further and assert that

$$\aleph_{\alpha+1} = \aleph_\alpha \otimes^{\aleph_0} \aleph_\alpha.$$

The reasonableness of this equation can be judged from the fact that the ordinal corresponding to $\aleph_1$, can be written in the form

$$\begin{aligned}\omega_1 &= \{0, 1, 2, \cdots \omega, \cdots \omega^2, \cdots \omega^\omega, \cdots {}^{\omega}\omega, \cdots, \cdots, \cdots\} \\ &= \{0, 1, 2, \cdots \omega, \cdots \omega \otimes^0 \omega, \cdots \omega \otimes^1 \omega, \cdots \omega \otimes^2 \omega, \cdots, \cdots, \cdots\}.\end{aligned}$$

This can be verified easily from the description of ω_1 given by Halmos earlier. One more equation we will assert is that

$$2^{\aleph_\alpha} = 2 \otimes^1 \aleph_\alpha.$$

With these notations we can state the axiom that we are interested in.

Axiom 1 (Axiom of Monotonicity). $\aleph_{\alpha+1} = \aleph_\alpha \otimes^{\aleph_0} \aleph_\alpha$, *and* $2^{\aleph_\alpha} = 2 \otimes^1 \aleph_\alpha$. *Further, if* $m_1 \leq m_2$, $k_1 \leq k_2$, *and* $n_1 \leq n_2$, *then* $m_1 \otimes^{k_1} n_1 \leq m_2 \otimes^{k_2} n_2$.

2.2. Continuum Theorem.

If we accept the axiom of monotonicity, a significant theorem follows.

Theorem 2.1 (Continuum Theorem). $\aleph_{\alpha+1} = m \otimes^k \aleph_\alpha$, *for finite* $m > 1, k > 0$.

Proof. A direct consequence of the axiom of monotonicity is that, for finite $m > 1$ and $k > 0$,

$$2^{\aleph_\alpha} = 2 \ \otimes^1 \aleph_\alpha \leq m \otimes^k \aleph_\alpha \leq \aleph_\alpha \otimes^{\aleph_0} \aleph_\alpha = \aleph_{\alpha+1}.$$

When we combine this with Cantor's result

$$\aleph_{\alpha+1} \leq 2^{\aleph_\alpha},$$

the theorem follows.

Theorem 2.2 (GCH). $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$.

Proof. If we put $m = 2$, $k = 1$ in the Continuum Theorem, we get

$$\aleph_{\alpha+1} = 2 \ \otimes^1 \aleph_\alpha = 2^{\aleph_\alpha},$$

making GCH a theorem.

Theorem 2.3 (Axiom of Choice)**.** *Cartesian product of nonempty sets will always be nonempty, even if the product is of an infinite family of sets.*

Proof. Since GCH implies AC and since we have already proved GCH, axiom of choice follows.

3. AXIOM OF FUSION

Before we can state the axiom of fusion, it is necessary to give a few definitions. The most significant definition here is that of a bonded class, a class from which only a distinguished element called maximal element can be identified by the axiom of choice.

3.1. The Big Picture. Figure 1 should help build up a mental picture for the definitions that follow. We want to imagine how an infinitesimal part of a unit interval looks like, when magnified $\aleph_0$ times. The red line in the figure is, perhaps, as good a representation as any for an infinitesimal, and we can imagine that $\aleph_0$ such infinitesimals constitute a unit interval. The age-old question about a point on the real line is, whether it is a tiny *iron filing* or a *steel ball*. According to our view here, it is both. The line (A, B) in the figure is the filing and B is the ball, with the clear understanding that these are only figments of our imagination and can never be palpable. This fact has been at the root cause of incessant quibbling among generations of mathematicians, which made one mathematician finally call the infinitesimals "ghosts of departed quantities".

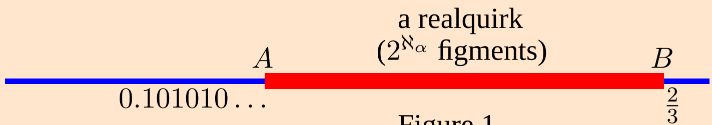

Figure 1

The binary sequence $0.1010101\ldots$ shown in the figure indicates that the infinitesimal in our visualization corresponds to the number $\frac{2}{3}$ in the unit interval. In the red line $(A, B]$, B is the element that can be identified and chosen by the axiom of choice and the rest of the $2^{\aleph_\alpha}$ elements in $(A, B]$ remains inaccessible even to the axiom of choice. For this reason, it may not be unreasonable to call (A, B) a *realquirk* and the elements in it *figments*. The essence of the Axiom of Fusion is that the elements of the infinitesimal $(A, B]$ are only figments of our imagination, except for B, which we may call a *realquark*.

3.2. Elements and Figments. With the bacground ideas and visualization given above, we can state our definitions.

Class: A set which has sets as its elements.
Family: A set which has classes as its elements.
Complete class: A class in which the union of the sets in it is also in the class. This distinguished element we will call the *maximal element* of the complete class.
Bonded class: A complete class from which the axiom of choice can choose only the maximal element and none else. The existence of such sets is what the Axiom of Fusion that follows is all about.
Realquark: The maximal element of a bonded class.
Figment: Any element of a bonded class other than the realquark.

Realquirk: The set containing all the elements of a bonded class other than the realquark.

$2^{\aleph_\alpha}$*:* Power set of $\aleph_\alpha$.

$\binom{\aleph_\alpha}{\aleph_\alpha}$*: Combinatorial set* of $\aleph_\alpha$, defined as the class of all the subsets of $\aleph_\alpha$ of cardinality $\aleph_\alpha$.

Bonded family: A family in which every element is a bonded class.

Real cardinality: The cardinality of a bonded family.

R*:* The class of *infinite recursive* subsets of positive integers, a class of cardinality $\aleph_0$.

x*:* An element of the class R, which defines an infinite binary sequence and hence equivalent to a real point in the unit interval $(0, 1]$.

$(x|$*:* The cartesian product $x \times 2^{\aleph_\alpha}$, which we will call the *infinitesimal* x.

Microcosm: The cartesian product $R \times 2^{\aleph_\alpha}$, considered as an adequate representation of all the points of the unit interval $(0, 1]$.

N: An element of R, which defines an infinite binary sequence written leftwards and hence called a *supernatural number*.

$|N)$: The cartesian product $2^{\aleph_\alpha} \times N$, and hence called a *cosmic stretch*.

Macrocosm: The cartesian product $2^{\aleph_\alpha} \times R$, considered as an adequate representation of all *counting numbers*, even those above supernatural numbers.

Using these definitions, we can now state the axiom of fusion.

Axiom 2 (Axiom of Fusion)**.** $(0, 1] = \binom{\aleph_\alpha}{\aleph_\alpha} = R \times 2^{\aleph_\alpha}$, *where* $x \times 2^{\aleph_\alpha}$ *is a bonded class.*

The axiom of fusion says that $(0,1]$ is a bonded family. Further, the significant *combinatorial* part of the power set of $\aleph_\alpha$ consists of $\aleph_0$ infinitesimal bonded classes, each of cardinality $2^{\aleph_\alpha}$. Thus the *real cardinality* of the family $\binom{\aleph_\alpha}{\aleph_\alpha}$ is $\aleph_0$.

We define Intuitive Set Theory as the theory we get when the axioms of monotonicity and fusion are added to ZF theory.

3.3. Unification Theorem. Some significant results follow from the axiom of fusion.

Theorem 3.1 (Combinatorial Theorem). $\binom{\aleph_\alpha}{\aleph_\alpha} = 2^{\aleph_\alpha}$.

Proof. A direct consequence of the axiom of fusion is that

$$2^{\aleph_\alpha} \leq \binom{\aleph_\alpha}{\aleph_\alpha}.$$

Since, $\binom{\aleph_\alpha}{\aleph_\alpha}$ is a subset of $2^{\aleph_\alpha}$,

$$\binom{\aleph_\alpha}{\aleph_\alpha} \leq 2^{\aleph_\alpha},$$

and the theorem follows.

Theorem 3.2 (Unification Theorem)**.** *All the three sequences*
$\aleph_0, \aleph_1, \aleph_2, \aleph_3, \ldots$
$\aleph_0, 2^{\aleph_0}, 2^{\aleph_1}, 2^{\aleph_2}, \ldots$
$\aleph_0, \binom{\aleph_0}{\aleph_0}, \binom{\aleph_1}{\aleph_1}, \binom{\aleph_2}{\aleph_2}, \ldots$
represent the same series of cardinals.

Proof. The axiom of monotonicity shows that the first two are the same, and the axiom of fusion shows that the last two are same.

Cantor's theorem asserts that every model of ZF theory has to have cardinality greater than $\aleph_0$. On the other hand, Löwenheim-Skolem theorem (LS) says that there is a model of ZF theory, whose cardinality is $\aleph_0$. These two contradictory statements together is called Skolem Paradox.

Intuitive set theory provides a reasonable way to resolve the Skolem Paradox. We merely take the LS theorem as stating that the *real* cardinality of a model of IST need not be greater than $\aleph_0$.

In ZF theory, it is known that there are sets which are not Lebesgue measurable, but it has not been possible to date to construct such a set, without invoking the axiom of choice. The usual method to produce a nonmeasurable set, is to choose exactly one element from each of the set $x \times 2^{\aleph_0}$ we defined earlier, and show that every one of the $2^{\aleph_0}$ sets thus created is not Lebesgue mesureable. This method is not possible in IST, because $x \times 2^{\aleph_0}$ is a bonded class and therefore the axiom of choice can choose only one element from $x \times 2^{\aleph_0}$, thereby, allowing it to produce only one set out of the $2^{\aleph_0}$ sets. Since the creation of all the $2^{\aleph_0}$ sets is crucial for the argument to establish the existence of nonmeasurable sets, we conclude that the axiom of choice cannot be used in IST for producing nonmeasurable sets. Hence, it would not be unreasonable to assert that there are no sets in IST which are not Lebesgue measurable.

4. CONCLUSION

From the definitions given above, it is obvious that there is a one-to-one correspondence between the points in $(0, 1]$ and the counting numbers. Hence, our statements about microcosm are equally applicable to macrocosm also. Further, it should be clear that intuitive set theory will suffice for scientists to investigate the phenomenal world, and classical set theory will be needed only if we want to probe the complexities of the noumenal universe.

The two axioms given here allow us to visualize the unit interval $(0, 1]$ in a simple way. We can consider $(0, 1]$ as a graph with $\aleph_0$ edges and $\aleph_0$ nodes, each edge representing a realquirk with $2^{\aleph_\alpha}$ figments in it. Each node represents a realquark corresponding to a *Dedekind cut* in the interval.

If the axioms of monotonicity and fusion do not produce any contradictions in IST, we can divide the statements of IST into four mutually exclusive categories: F is a *theorem*, if a proof exists for F, but not for $\overline{F}$. F is a *falsehood*, if a proof exists for $\overline{F}$, but not for F. F is an *introversion*, if a proof exists for $\overline{F}$ when F is assumed, and a proof for F exists when $\overline{F}$ is assumed. F is a *profundity*, if a proof exists for neither F nor $\overline{F}$, and it is not an introversion.

It is easy to see that an introversion cannot be chosen as an axiom, since it will surely create a contradiction in the theory. Gödel has shown that a consistency statement in any theory is an introversion. The conclusion is that even though we might believe in the consistency of a theory, we can never choose it as an axiom. Note that according to our definitions, generalized continuum hypothesis and axiom of choice are profundities in ZF theory, whereas they are theorems in IST.

The main problem of mathematics is to classify the *entire set* of formulas of IST into the four categories above. A great achievement of the twentieth century is the recognition that this is never possible.

EXERCISES

EXERCISE 1. Evaluate $2 \otimes^3 4$ and compare it with 136×2^{256}, the number of electrons in the universe, as given by Eddington.

EXERCISE 2. Prove or disprove that $2 \otimes^3 n + 1$ is a prime, for all positive integer values of n.

EXERCISE 3. Show that $m \otimes^k n$ gives the solution to Ackermann Equations.

[1] K. K. Nambiar, *Real Set Theory*, Computers and Mathematics, Vol. 38, No. 7-8 (1999), pp. 167-171.

[2] K. K. Nambiar, *Intuitive Set Theory*, Computers and Mathematics, Vol. 39, No. 1-2 (2000), pp. 183-185.

[3] P. Benacerraf, and H Putnam, eds., *Philosophy of Mathematics: Selected Readings*, Prentice-Hall, Englewood Cliffs, NJ, 1964.

[4] P. R. Halmos, *Naive Set Theory*, D. Van Nostrand Company, New York, NY, 1960.

Exercise 1. A detailed evaluation shows that

$$\begin{aligned} 2 \otimes^3 4 &= 2 \otimes^2 [2 \otimes^2 [2 \otimes^2 2]]] \\ &= 2 \otimes^2 [2 \otimes^2 4] \\ &= 2 \otimes^2 65536 \\ &= 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}, \end{aligned}$$

where the total number of 2's tilting forward is 65536. Obviously, there is no comparison between $2 \otimes^3 4$ and the number of electrons in the universe. Exercise 1

Exercise 2. It took a fortunate Euler to disprove Fermat's conjecture that

$$2^{2^n} + 1$$

is a prime, for all n. It turns out that $2^{2^5} + 1$ has a factor 641. The following shows what we are up against.

$$\begin{aligned}
2 \otimes^3 1 + 1 &= 2 + 1 \\
&= 3, \text{ a prime} \\
2 \otimes^3 2 + 1 &= 2^2 + 1 \\
&= 4 + 1 \\
&= 5, \text{ a prime} \\
2 \otimes^3 3 + 1 &= 2^{2^{2^2}} + 1 \\
&= 65536 + 1 \\
&= 65537, \text{ a prime}
\end{aligned}$$

Since,

$$2 \otimes^3 4 + 1$$

is unimaginably large, as shown ealier, we will never be able to decide for sure whether it has a factor. A similar statement applies to

$$2 \otimes^3 n + 1,$$

for $n > 4$. In short, we cannot decide the conjecture. Exercise 2

Exercise 3. Ackermann functions are defined by the equations,

$$\begin{aligned} A(0,n) &= mn \\ A(k,1) &= m \\ A(k,n) &= A[k-1, A(k,n-1)]. \end{aligned}$$

Comparing this with the definition of $\otimes^k$ shows that there is virtually no difference between the two.

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